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ELEMENTARY MECHANISM:

A Text-Book

FOR STUDENTS OF MECHANICAL ENGINEERING.

BY

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PREFACE.

Quite a number of treatises have appeared on the subject of Kinematics, or Pure Mechanism, most of which are now in print, so that a few words of explanation as to the reasons for publishing this book seem necessary.

In searching for a text-book on this subject for the use of our classes of Mechanical Engineering students, we were unable to find a book which met our requirements. Some were so vague and incomplete as to be almost useless, while others were large, exhaustive treatises, more valuable as books of reference than as text-books for the use of students. The following pages were therefore prepared in the form of lectures; the object being, to give a clear description of those mechanical movements which may be of practical use, together with the discussion of the principles upon which they depend. At the same time, all purely theoretical discussions were avoided, except where a direct practical result could be reached by their introduction. These lectures were used in our classes; and, having proved comparatively satisfactory in that shape, it was thought best to publish them, after making such improvements as our class-room experience dictated.

We make little claim to originality of subject-matter, free use having been made of all available matter bearing on the subject. There is, in fact, very little room for such originality, the ground having been almost completely covered by previous writers. Our claim to consideration is based almost entirely on the manner in which the subject has been presented. Accuracy, clearness, and conciseness are the points that we have tried to keep constantly in view. While much has been omitted that is of merely abstract interest, yet it is believed that nearly all that is of direct practical importance will be found in these pages.

We have, in common with nearly all other writers on this subject, closely followed the general plan of Professor Willis' "Principles of Mechanism." Other works which have been consulted and to which we are indebted are Rankine's "Machinery and Mill work," Reuleaux' "Le Constructeur," and Goodeve's "Elements of Mechanism;" and in a less degree, Belanger's "Cinematique," Reuleaux' "Kinematics of Machinery," Robinson's "Teeth of Wheels," Grant's "Teeth of Gears," Appleton's "Cyclopædia of Mechanics," and Unwin's "Elements of Machine Design."

CONTENTS.

CHAPTER I.

PAGE

INTRODUCTION 1

CHAPTER II.

ELEMENTARY PROPOSITIONS 10

Graphic Representation of Motion. — Composition and Resolution of Motions. — Modes of transmitting Motion. — Velocity Ratio. — Directional Relation.

CHAPTER III.

COMMUNICATION OF MOTION BY ROLLING CONTACT. — VELOCITY RATIO CONSTANT. — DIRECTIONAL RELATION CONSTANT 27

Cylinders. — Cones. — Hyperboloids. — Practical Applications. — Classification of Gearing.

CHAPTER IV.

COMMUNICATION OF MOTION BY ROLLING CONTACT. — VELOCITY RATIO VARYING. — DIRECTIONAL RELATION CONSTANT 52

Logarithmic Spirals. — Ellipses. — Lobed Wheels. — Intermittent Motion. — Mangle Wheels.

CHAPTER V.

COMMUNICATION OF MOTION BY SLIDING CONTACT. — VELOCITY RATIO CONSTANT. — DIRECTIONAL RELATION CONSTANT. — TEETH OF WHEELS 66

Special Curves. — Rectification of Circular Arcs. — Construction of Special Curves. — Circular Pitch. — Diametral Pitch.

CHAPTER VI.

PAGE

COMMUNICATION OF MOTION BY SLIDING CONTACT.—VELOCITY RATIO CONSTANT. — DIRECTIONAL RELATION CONSTANT. — TEETH OF WHEELS (*Continued*) 85

Definitions.—Angle and Arc of Action.—Epicycloidal System.—Interchangeable Wheels.—Annular Wheels.—Customary Dimensions.—Involute System.

CHAPTER VII.

COMMUNICATION OF MOTION BY SLIDING CONTACT.—VELOCITY RATIO CONSTANT. — DIRECTIONAL RELATION CONSTANT. — TEETH OF WHEELS (*Continued*) 113

Approximate Forms of Teeth.—Willis' Method.—Willis' Odontograph. — Grant's Odontograph. — Robinson's Odontograph.

CHAPTER VIII.

COMMUNICATION OF MOTION BY SLIDING CONTACT.—VELOCITY RATIO CONSTANT. — DIRECTIONAL RELATION CONSTANT. — TEETH OF WHEELS (*Concluded*) 131

Pin Gearing.— Low-numbered Pinions.— Unsymmetrical Teeth. — Twisted Gearing. — Non-circular Wheels. — Bevel Gearing. — Skew-bevel Gearing. — Face Gearing.

CHAPTER IX.

COMMUNICATION OF MOTION BY SLIDING CONTACT. — VELOCITY RATIO AND DIRECTIONAL RELATION CONSTANT OR VARYING 165

Cams.— Endless Screw. — Slotted Link. — Whitworth's Quick Return Motion. — Oldham's Coupling. — Escapements.

CHAPTER X.

COMMUNICATION OF MOTION BY LINKWORK. — VELOCITY RATIO AND DIRECTIONAL RELATION CONSTANT OR VARYING 193

Classification. — Discussion of Various Classes. — Quick Return Motion. — Hooke's Coupling. — Intermittent Linkwork. — Ratchet Wheels.

CHAPTER XI.

COMMUNICATION OF MOTION BY WRAPPING CONNECTORS.—VELOCITY RATIO CONSTANT.—DIRECTIONAL RELATION CONSTANT 227

Forms of Connectors and Pulleys.—Guide Pulleys.—Twisted Belts.—Length of Belts.

CHAPTER XII.

PAGE
TRAINS OF MECHANISM 240
Value of a Train. — Directional Relation in Trains. — Clockwork. — Notation. — Method of Designing Trains. — Approximate Numbers for Trains.

CHAPTER XIII.

AGGREGATE COMBINATIONS 263
Differential Pulley. — Differential Screw Feed Motions. — Epicyclic Trains. — Parallel Motions. — Trammel. — Oval Chuck.

PROBLEMS 289

INDEX 295

ELEMENTARY MECHANISM.

CHAPTER I.

INTRODUCTION.

1. A Machine is a combination of fixed and movable parts, interposed between the power and the work for the purpose of adapting the one to the other.

This definition presupposes the existence of two things; namely, a source of power, and a certain object to be accomplished. The source of power may be one of the forces of nature applied directly, such as the expansive force of steam in a steam engine, or it may be obtained by the indirect application of such natural forces; that is, the latter may have been already modified by some other machine. Thus, when a steam engine drives the machinery of a shop by means of a line of shafting, the latter may properly be considered as the source of power of the individual machine.

2. Mechanism.—In designing a machine, we must take into consideration both the motions to be produced and the forces to be transmitted. But these two elements may most conveniently be discussed and investigated separately; and such discussions and investigations constitute the two divisions of the general subject of mechanism; namely, Pure Mechanism and Constructive Mechanism.

Pure Mechanism, then, treats of the designing of machines, as far as relates to the transmission and modification of motion, and explicitly excludes all considerations of force transmitted, or of strength and durability of parts.

In order that the sense in which we shall use certain fundamental terms may be clearly understood, we shall now give an explanation of such words and phrases.

3. Motion and Rest. — These terms are essentially relative. When a body changes its position with regard to some fixed point, it is said to be *in motion* relatively to that point; when no such change is taking place, it is said to be *at rest* relatively to that point. Two bodies may evidently be in motion relatively to a third, and still be at rest with regard to each other.

4. Path. — When a point moves from one position to another, it describes a line, either straight or curved, connecting the two positions. This line is called its *path*. But the path alone does not completely define the motion, for the point may move in the path in either of two *directions;* as, up or down, to the right or to the left, in the direction of the hands of a watch or the reverse.

5. Kinds of Motion. — Motion may take place along either a straight or curved path; in the former case it is termed *rectilinear* motion, and in the latter case *curvilinear* motion. In either case, when a moving point travels forward and backward over the same path, it is said to have a *reciprocating* motion. For example, the piston of a locomotive has reciprocating rectilinear motion. In the particular case where the reciprocating point moves in the arc of a circle, as, for example, the weight of a pendulum, it is said to *oscillate*, or, by some, to *vibrate*. When the motion of a point is interrupted by certain definite intervals of rest, it is said to have an *intermittent* motion. The motion of the escape wheel of a clock is of this kind.

6. Revolution and Rotation. — These terms are ordinarily used synonymously, to denote the turning of a body about an axis; and no ambiguity is usually likely to arise from so using them. Thus, the fly wheel of an engine is said to rotate or revolve. By more strict definition, *rotation* should be applied only to the turning of a body about an axis which passes through it, while *revolution* is a more general term to include the motion of a body along a path which is a closed curve. Thus, the earth *rotates* about its axis and *revolves* about the sun.

7. Velocity. — In addition to the path and direction of a moving body, there is another element necessary to completely determine its motion, and that is its *velocity.*

Velocity is measured by the relation between the *distance* passed over and the *time* occupied in traversing that distance. Velocity may be uniform and unchanging, or it may become greater or less; and then changes may take place quickly or slowly, regularly or irregularly. But, for our purposes, it is sufficient to consider only two kinds of velocity, *constant* or *uniform*, and *variable.*

Velocity is expressed numerically by the number of units of distance passed over in one unit of time. The units of distance and time may be selected at pleasure; but, for mechanical purposes, the most convenient units are feet and minutes; and these will, in general, be employed throughout this volume.

When a body moves with a *uniform* velocity, the distance passed over varies directly with the time. Thus, if by V we designate the velocity, and by S the total distance passed over in the time T, we have $S = VT$.

Again: if the velocity is given, we may find the time T to traverse a given distance S, for $T = \frac{S}{V}$.

When the distance and the time are given, we may determine the velocity from the equation $V = \frac{S}{T}$.

For example, if a body moves at a uniform velocity over a distance of 100 feet, and occupies 5 minutes in doing so, it has a velocity $V = \frac{S}{T} = \frac{100}{5} = 20$ feet per minute.

In case the velocity is *variable*, these expressions do not give the velocity at any particular instant, but only the *mean* velocity for the whole time considered. The velocity at any particular instant is measured by the distance which the body would pass over in the next succeeding unit of time, were the velocity with which the body commences that unit to continue uniformly throughout it. Thus, if a railway train is slowing down in coming to a stop, its velocity is decreasing, but may, nevertheless, be measured at any instant. If, for instance, we say that the train has a velocity of 20 miles per hour, we mean, that, if it were to continue in motion for one hour at the velocity which it has at that instant, it would travel 20 miles.

8. Angular Velocity. — The most natural way of expressing the velocity of a rotating body consists in stating the angle through which it turns, or the number of revolutions which it makes, in the unit of time. When the number of revolutions is given, it must usually be expressed as an angle before it can be used in calculation; and the angle may be stated in degrees or in circular measure. For convenience of comparison with linear velocities, we shall define *angular velocity* to be the velocity of a rotating body thus expressed in circular measure; i.e., as the quotient obtained by dividing the length of the arc subtending the angle through which it turns in one unit of time, by the length of the radius of that arc. All the points of a rotating body move with the same angular velocity, but the linear velocity of each point varies directly with its radial distance from the centre of motion.

Let a = angular velocity of a body, R = radial distance of some point in that body, and V = linear velocity of that point; in other words, the length of the arc which it describes in the unit of time.

Then $$a = \frac{V}{R}.$$

Thus, if a locomotive having driving-wheels 5 feet in diameter is moving at a speed of 30 miles an hour, the linear velocity of a point on the rim of the wheel, relatively to the frame of the engine, is evidently $V = \frac{S}{T} = \frac{30 \times 5280}{60}$ $= 2640$ feet per minute. The angular velocity of the wheel is therefore $a = \frac{V}{R} = \frac{2640}{2\frac{1}{2}} = 1056$ feet per minute.

The relation between the number of revolutions per minute and the angular velocity is readily found. Thus, let a wheel make N revolutions in T minutes. Let a point be taken at a radial distance R. Then this point will, in each complete revolution, describe a circle whose length is $2\pi R$; in T minutes it will describe N such circles, and travel a distance $2\pi NR$, and its linear velocity $V = \frac{2\pi NR}{T}$. Hence its angular velocity is $a = \frac{V}{R} = \frac{2\pi NR}{TR} = \frac{2\pi N}{T}$. When T is unity, that is, when N is the number of revolutions per minute, $a = 2\pi N$, and $N = \frac{a}{2\pi}$. Hence, in the above example of the locomotive driving-wheel, we find that the wheel makes $N = \frac{a}{2\pi} = \frac{1056}{2 \times 3.1416} = 168.07$ revolutions per minute.

9. Periodic Motion. — During the operation of a machine, it usually happens that the various moving parts go through a series of changes of motion which recur perpetually in the same order. The interval of time which includes in itself one such complete series of changes is called a *period*, and the character of the motion is described by the term *periodic*. The complete series of changes of motion included in one period is called a *cycle*.

In periodic motion, the general law of the succession of changes is the same in successive periods, but the actual time may vary; that is, the periods may be unequal in length. As a rule, however, the periods are equal, and the duration, magnitude, and law of succession of the changes are identical, in successive periods; such motion is known as *uniform periodic motion*.

10. Classification of Parts of Machines. — As the work for which machines are designed varies so widely, and as they may be actuated by so many different kinds of power, we find great differences in them as to details of construction and manner of operation. But, in spite of these differences, every machine may be considered to consist of *three* classes of parts. At one end we have the parts which are specially designed to receive the action of the power; at the other we have those which are determined in form, position, and motion, by the nature of the work to be done. Between and connecting the former and the latter, we find the parts which are interposed for the purpose of transmitting and modifying the force and the motion; so that, when the first parts move according to the law assigned them by the action of the power, the second must necessarily move according to the law required by the character of the work. These three classes of parts are so far independent of one another, that any kind of work may be done by any kind of power, and by means of various combinations of interposed mechanism. The motion of the parts which receive the action of the power must be transmitted to the working-parts; and, as the action of the latter is usually very different from that of the former, it follows that the motion must be modified, during transmission, according to certain definite conditions.

This modification is accomplished by means of the interposed mechanism above mentioned, and it is to the discussion of the methods by which motion may be transmitted and modified that the following pages are devoted.

11. A Train of Mechanism is composed of a series of movable pieces, each of which is so connected with the frame-work of the machine, that when in motion every point of it is constrained to move in a certain path, in which, however, if considered separately from the other pieces, it is at liberty to move in the two opposite directions, and with any velocity. Thus, wheels, pulleys, shafts, and rotating pieces generally, are so connected with the frame of the machine, that any given point is compelled, when in motion, to describe a circle round the axis, and in a plane perpendicular to it. Sliding pieces are compelled by fixed guides to describe straight lines, other pieces to move so that their points describe more complicated paths, and so on.

12. These pieces are connected in successive order in various ways so that when the first piece in the series is moved from any external cause, it compels the second to move, which again gives motion to the third, and so on. The variation in the laws of motion of the different pieces of a train is effected by the mode of connection.

13. Modes of Connection. — One piece may transmit motion to another by *direct contact*, or by means of an *intermediate connector*. In the latter mode of connection, the motion of the intermediate piece is usually of no importance, the object to be secured being simply the proper relative motion of the two primary pieces. Two pieces connected in either of the above ways, so that a definite motion of one of the pieces will produce an equally definite motion of the other, form an *elementary combination*. A train of mechanism evidently consists of a series of such elementary combinations, each piece receiving motion from the one that precedes it, and transmitting motion to the one next in order.

14. That piece of an elementary combination to which motion is imparted from some extraneous source is termed the *driver;* and that piece whose motion is received from, and governed by, the driver is termed the *follower.*

15. Velocity Ratio and Directional Relation. — It has been already shown that the paths of the pieces in an elementary combination are fixed, and depend on the connection of the pieces with the framework of the machine; while their velocity and direction of motion may vary, and must be determined for each instant of action. Thus, in comparing the motions of the pieces for successive instants, we may find changes of velocity or of direction, or both. But, while the absolute velocities and the absolute directions of both pieces may be liable to continual variation, it is evident that there will exist, at each instant, a certain definite ratio between the velocities, and an equally definite relation between the directions, of the driver and follower. This *velocity ratio* and this *directional relation* will depend solely on the manner in which the two pieces are connected, and will be entirely independent of their absolute velocities or directions. The velocity ratio, and also the directional relation, may be *constant* during the entire period, or either or both may *vary.* For example, if two circular wheels turning on fixed axes gear with each other, their velocity ratio is constant. If one wheel is twice as large as the other, it will make only one-half as many turns in the same time, or its angular velocity will be half that of the smaller wheel. But during any changes in velocity whatsoever, as one wheel cannot rotate without turning the other, and as the respective radii of contact do not change in length, the ratio of their velocities at any instant is the same; that is, such wheels have a *constant* velocity ratio. And so, also, of the relative directions of the rotations. If the wheels are in external gear, they will turn in opposite directions; if in internal gear, in the same direction: but in either case the directional relation will remain *constant*, without regard to any change of absolute direction of the driver.

If the two wheels are elliptical, however, as those shown in Fig. 42, the directional relation will be *constant*, while the velocity ratio will *vary* according to the varying lengths of the radii of contact.

If, then, in addition to the paths of both driver and follower, we have determined their velocity ratio, and the directional relation of their motion, for every instant of an entire period, our knowledge of the action of the combination will be complete.

CHAPTER II.

ELEMENTARY PROPOSITIONS.

Graphic Representation of Motion. — Composition and Resolution of Motions. — Modes of Transmitting Motion. — Velocity Ratio. — Directional Relation.

16. Graphic Representation of Motion. — The problems relating to the motions of points may be most readily solved by geometrical construction. It is evident that the rectilinear motion of a point may be represented by a straight line; for the *direction* of the line may represent the direction of the motion, while the velocity may be indicated by its *length.* When a point moves in a curve, its direction of motion at any instant is the same as the direction of the tangent to the curve at the point considered. Hence the curvilinear motion of a point may be represented in the same manner as the rectilinear motion, using the direction of the tangent as the direction of the straight line above mentioned, and making its length proportional to the velocity, as before.

By thus representing the motion of properly selected points, we may establish certain relations, by purely geometrical reasoning, which will not only enable us to obtain the velocity ratio and the directional relation in the particular phase represented, but will lead to, and almost involve, the accurate construction on paper of the movements considered. The latter is such an important advantage in practical work, that this method is greatly to be preferred, and has been adopted in this volume.

17. Composition of Motions. — If a material point receives a single impulse in a given direction, it will move in that direction with a certain velocity; and, as above explained, its motion may be represented by a straight line having the same direction as the motion, and of a length proportional to the velocity. If a point receives, at the same time, two impulses in different directions, it will obey both, and move in an intermediate direction with a velocity differing from that due to either impulse alone. Such a point may receive, at the same instant, any number of impulses, each one tending to impart to it a motion in a definite direction and with a certain velocity. Now, it is evident that the point can move only in one direction and with one velocity; this motion is called the *resultant;* and the separate motions which the different impulses, taken singly, tended to give it, are called the *components.*

18. Parallelogram of Motions. — Given two component motions of a point, to find the resultant.

In Fig. 1, let the point A be acted on at the same time by two impulses, tending to give it the motions represented, in direction and velocity, by the straight lines AB and AD respectively. Through B draw BC parallel to AD; through D draw DC parallel to AB; join AC. Then AC will represent, in direction and velocity, the motion which the point A will have as the result of the two impulses which separately would have produced the motions AB and AD respectively. The length of the resultant may be altered by varying the lengths of the components or the angle between them, but in no case can it exceed their sum nor be less than their difference.

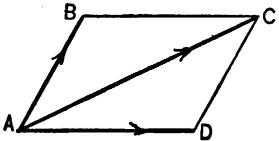

Fig. 1

This proposition is known as the *parallelogram of motions,* and may be thus stated: —

If two component motions be represented, in direction and velocity, by the adjacent sides of a parallelogram, the resultant will be similarly represented by the diagonal passing through their point of intersection.

19. Polygon of Motions.—By a repetition of the above process, we may find the resultant of any number of simultaneous independent components.

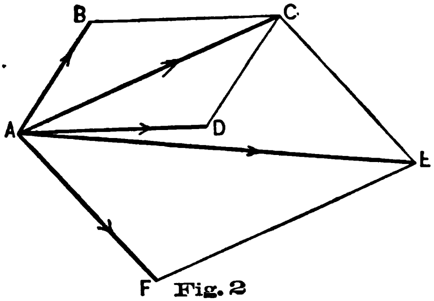

Fig. 2

In Fig. 2, let *AB*, *AD*, *AF*, represent three such components. We first compound any two of them, as *AB* and *AD*, by completing the parallelogram *ABCD*, and find the resultant *AC*. We next compound *AC* with *AF* in a similar manner, and find the resultant *AE*. The latter is evidently the resultant of the *three* components.

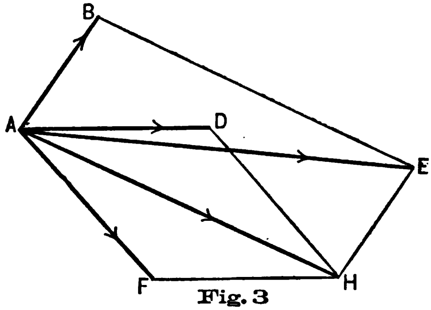

Fig. 3

This process may be continued for any number of components, and it makes no difference in what order they are taken. In Fig. 3, for instance, we have the same components as in Fig. 2, and find the same resultant, *AE*, though the composition is carried on in a different order.

20. Resolution of Motion. — This is the inverse of the process just explained. It is obvious, that, if two or more independent motions can be compounded into a single equivalent motion or resultant, the latter can be again separated, or *resolved*, into its components. But it evidently makes no difference whether the single motion to be resolved is the resultant of a previous composition, or whether it is an original independent motion. Any single motion can be resolved into two others, each of these again into two others, and so on as far as desired; these components being given any directions at pleasure. In Fig. 4, let *AC* represent the given motion. Through *A* draw the indefinite lines *AE* and *AH* in the directions in which it is desired to resolve *AC*. Through *C* draw *CB* parallel to *AH*, and intersecting *AE* at *B*; also *CD* parallel to *AE*, and intersecting *AH* at *D*. Then *AB* and *AD* will be the components required; and it is evident that by their composition (Art. 18) we would find their resultant to be *AC*, the given motion.

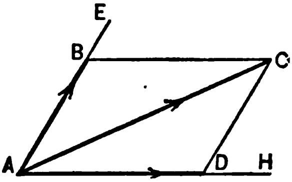

Fig. 4.

21. Communication of Motion by Direct Contact. — In Fig. 5, let *AD* and *BC* be two successive pieces of a train of mechanism, turning about the centres *A* and *B* respectively. Let *AD* be the driver, turning the follower *BC*, by contact, between the curved edges, as shown. Let *c* be the point of contact between the two pieces; and let the driver move the follower, until they occupy the positions shown by dotted lines, the points *a* and *b* having come in contact at *d*. During this motion, every point of the curved edge of the follower between *b* and *c* has been in contact with some point of the curved edge of the driver between *a* and *c*. If *bc* is not equal in length to *ac*, it is evident that sliding of one edge on the other must have taken place through a space equal to their difference; but, if $bc = ac$, there will have been no sliding. In the latter case the motion is said to be communicated by *rolling* contact, and in the former case by *sliding* contact.*

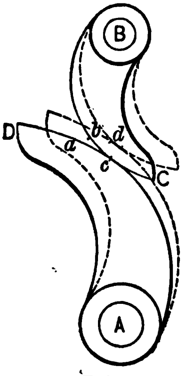

Fig. 5

Motion, then, may be communicated by two kinds of direct contact:—

1. By *rolling* contact, when each point of contact of the driver with the follower is continually changed, but so that the curve joining any given pair of points of contact of the driver shall be equal in length to the curve joining the respective points of the follower.

2. By *sliding* contact, when each point of contact of the driver with the follower is continually changed, but so that the curve joining any given pair of points of contact of the driver shall *not* be equal in length to the curve joining the respective points of the follower.

In contact motions, one or both of the curves must be convex; and, in the former case, the convex edge must have a sharper curvature than the concave edge. If this condition is not fulfilled, contact will take place at discontinuous points.

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* More strictly speaking, sliding contact should be defined as that motion in which every point of contact of one piece comes into contact with all the consecutive points, in their order, of a line in the other. Thus, the piston of a steam engine moves in true sliding contact with the interior surface of the cylinder. When this definition of sliding contact is adopted, it is usual to class under the head of mixed contact those contact motions which partake of both rolling and sliding. But, for our purposes, it is sufficient to distinguish between contact which is rolling and that which is not; designating by the term "sliding" not only that which is purely so, as just defined, but also the cases just spoken of as mixed contact.

22. Communication of Motion by Intermediate Connectors. — Such intermediate connectors may be divided into two general classes: *links*, which are rigid, and must be jointed or pivoted to both the driver and follower; and *bands*, or *wrapping connectors*, which are flexible.

The former class includes all forms of rigid connectors which can transmit motion by pushing or pulling, such as connecting-rods, locomotive side rods, etc.; the latter includes all forms of connectors which can transmit motion by pulling only, such as belts, ropes, chains, etc.

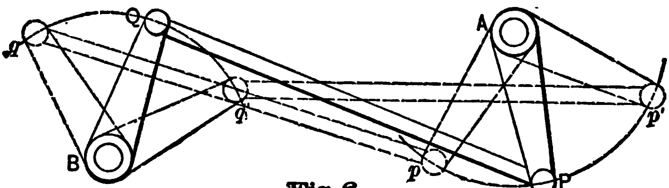


Fig. 6

In Fig. 6, let *AP*, *BQ*, be driver and follower, moving about the centres *A* and *B* respectively, and connected by the link *PQ*. If *AP* is turned so as to occupy another position, *Ap* or *Ap′*, it will, by means of the link, move the arm *BQ* into the position *Bq*, or *Bq′*. If the driver *push* the follower, the connector is necessarily rigid, and, as just stated, belongs to the general class of *links*. But the connector may be flexible, as in Fig. 7, where *ACE* is the driver, and *BDF* the follower, turning about the centres *A* and *B* respectively, and connected by a flexible but inextensible band which lies in the direction of the common tangent of the two curves. If the driver be moved in the direction of the arrow, it will, by means of this connector, turn the follower as indicated; and the connector will unwrap itself from the curved edge of the latter, and wrap itself on that of the former. By means of this form of intermediate connector, which belongs to the general class of *bands* or *wrapping connectors*, it is evident that motion can be transmitted by *pulling* or *tension* only.

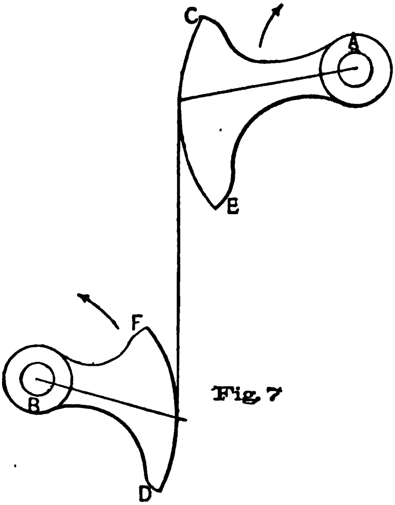


Fig. 7

23. Modes of Transmission of Motion. — Every elementary combination may be classified according to one of the four modes of transmission of motion just defined; namely, —

1. Rolling contact.	3. Linkwork.
2. Sliding contact.	4. Wrapping connectors.

24. Velocity Ratio in Linkwork. — In Figs. 8 and 9, let AP, BQ, be two arms, turning on fixed centres A and B respectively, and connected by the rigid link PQ. Since the arm AP turns about the centre A, the point P will move in the arc of a circle, and hence its direction of motion at any instant will be represented by the tangent to that arc; that is, by a line perpendicular to the radius AP. Draw Pa perpendicular to AP, and of such a length as to represent the velocity V of the point P in that direction at that instant. Similarly, draw Qb perpendicular to BQ to represent, by length and direction, the velocity v of the point Q at that instant. Let fall the perpendiculars AN and BM from the fixed centres of motion upon the line of the link; and let T be the point of intersection of the line of the link with the line of centres.

The resolved velocity of P along the line of the link is $V \cos aPc = V \cos PAN = V \cos \theta$; and the resolved velocity of Q along the line of the link is $v \cos bQd = v \cos QBM = v \cos \phi$. Since the link PQ is rigid, and can be neither extended nor compressed, the resolved velocities of the points P and Q along the line of the link must be equal. Hence,

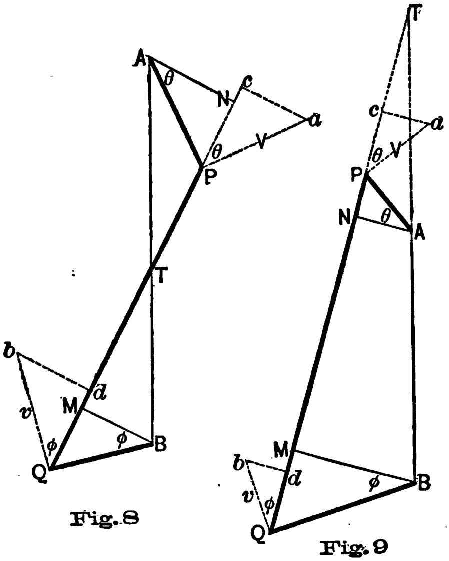


Fig. 8

Fig. 9

$$V \cos \theta = v \cos \phi. \tag{1}$$

$$\frac{v}{V} = \frac{\cos \theta}{\cos \phi}. \tag{2}$$

Let α = angular velocity of P about the centre A; and
α' = angular velocity of Q about the centre B;
then (Art. 8),

$$\alpha = \frac{V}{PA}; \quad \alpha' = \frac{v}{QB},$$

$$\frac{\alpha'}{\alpha} = \frac{PA}{QB} \times \frac{v}{V} = \frac{PA \cos \theta}{QB \cos \phi} = \frac{AN}{BM}. \tag{3}$$

Also, since angle ATN = angle BTM, we have,

$$\frac{a'}{a} = \frac{AN}{BM} = \frac{AT}{BT}. \tag{4}$$

Hence, in the communication of motion by linkwork, —

1. The angular velocities of the arms are inversely proportional to the perpendiculars from the fixed centres of motion upon the line of the link.

2. The angular velocities of the arms are inversely proportional to the segments into which the line of the link divides the line of centres.

25. This proposition may also be proved by means of the *instantaneous centre.*

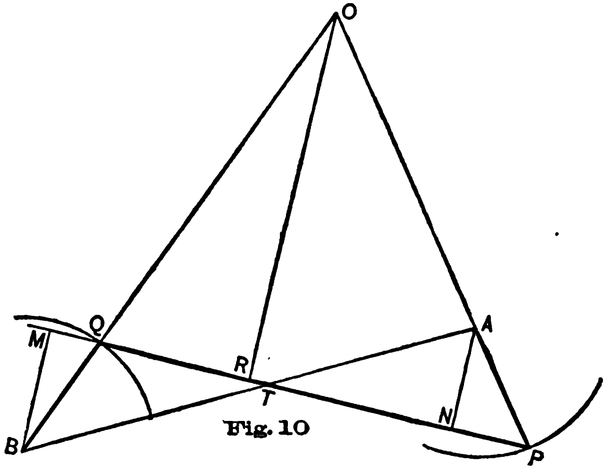


Fig. 10

In Figs. 10 and 11 the link PQ may be regarded as turning, during each instant of its motion, about some centre in space. This centre may be constantly changing its position in space, and also with regard to the line PQ itself; but at any given instant every point in PQ has the same angular velocity about this centre, and moves in a direction perpendicular to the line joining it to the centre, and with a linear velocity proportional to its distance from it. As *P* moves perpendicularly to *AP*, the centre must lie in *AP* (produced if necessary); and as *Q* moves perpendicularly to *BQ*, it must also lie in *BQ* (produced if necessary): hence it will be found at the intersection of these two lines at *O*. Let *V* and *v* represent the linear velocities of *P* and *Q* respectively. As both *P* and *Q* have the same *angular* velocity about O, their *linear* velocities will be proportional to their distance from that point; that is,

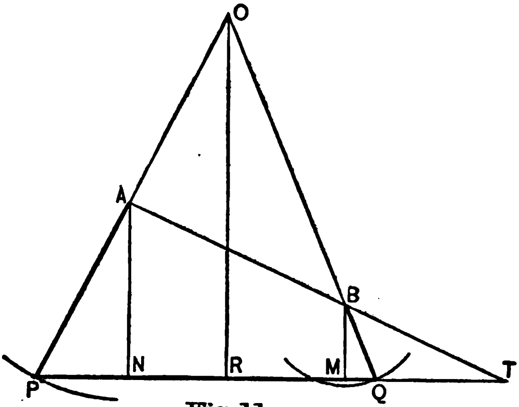


Fig. 11

$$V : v :: PO : QO.$$

Let a and a' be the angular velocities of *P* and *Q* about *A* and *B* respectively. Then

$$a : a' :: \frac{V}{AP} : \frac{v}{BQ} :: \frac{PO}{AP} : \frac{QO}{BQ}.$$

Let fall the perpendiculars OR, AN, and BM upon the line of the link; then, from the similar triangles ANP and ORP, BQM and OQR, BTM and ATN, we have

$$\frac{PO}{AP} = \frac{OR}{AN} \quad \text{and} \quad \frac{QO}{BQ} = \frac{OR}{BM}.$$

Hence

$$\frac{a'}{a} = \frac{OR}{BM} \times \frac{AN}{OR} = \frac{AN}{BM} = \frac{AT}{BT},$$

as before.

26. Directional Relation.—From Figs. 8, 9, 10, and 11, it is evident that the directional relation of the rotations of the two arms depends on the position of the centres A and B with reference to the line of the link PQ. If they are on the same side of PQ, the rotations will take place in the same direction; if on opposite sides, the rotations will be in contrary directions.

27. Velocity Ratio in Wrapping Connectors.—In Figs. 12 and 13, let AC and BD be two curved pieces moving about the fixed centres A and B respectively, and let them be connected by the flexible but inextensible band $EPQF$, fastened to them at E and F. If AC be turned in the direction of the arrow, it will cause BD to turn by means of the band, which will unwrap itself from the curved edge of BD, and wrap itself on that of AC. Let P and Q be the points at which the line of the band is tangent to the

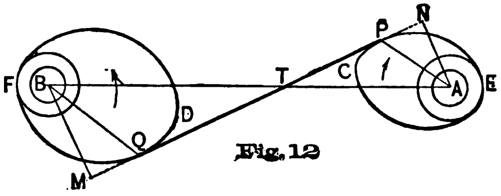


Fig. 12

curved edges. These points must move perpendicularly to the radii AP and BQ; and the action at any instant is precisely the same as that of two arms, AP and BQ, connected by a link, PQ, as discussed in the preceding articles. Hence, letting fall the perpendiculars AN and BM upon the common tangent, which is the line of the wrapping-connector, and finding the intersection T of the latter and the line of centres, it follows, that, in the communication of motion by wrapping connectors, —

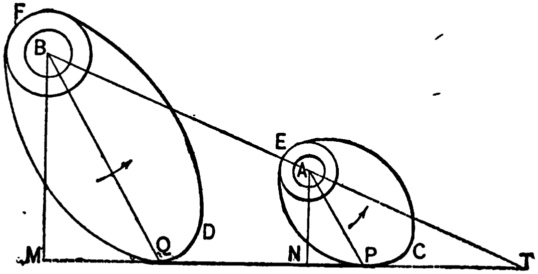


Fig. 13

1. The angular velocities of the pieces are inversely proportional to the perpendiculars from the fixed centres of motion upon the line of the wrapping connector.

2. The angular velocities of the pieces are inversely proportional to the segments into which the line of the wrapping connector divides the line of centres.

28. Directional Relation. — From Figs. 12 and 13, it is evident that the directional relation of the rotations of the two pieces depends on the position of the centres A and B with reference to the line of the wrapping connector PQ. If they are on the same side of PQ, the rotations will take place in the same direction; if on opposite sides, the rotations will take place in contrary directions.

29. Velocity Ratio in Contact Motions. — In Figs. 14 and 15 let the sectioned portions in contact at p represent parts of two curved pieces turning about fixed centres A and B. These curved edges may both be convex, as in the figures; or one may be concave, provided that the curvature of the convex edge is sharper than that of the concave edge.

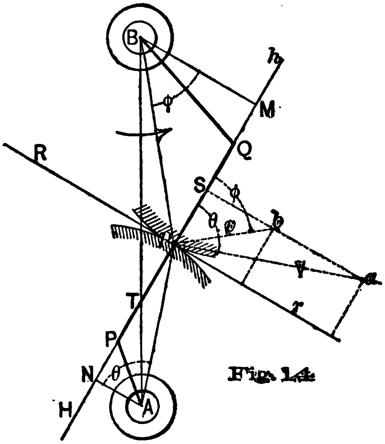


Fig. 14.

If the lower piece be turned in the direction of the arrow, it will drive the upper piece, and compel it to turn as indicated.

Draw Hh, the common normal, and Rr, the common tangent, to the curves at the point of contact.

The point p of the lower piece moves at any instant in a direction perpendicular to the radius Ap. Draw pa perpendicular to Ap, and of such a length as to represent the velocity V of the point p in that direction at that instant. Similarly, let pb drawn perpendicular to Bp represent the velocity v of the point p of the upper piece at that instant. Resolving these velocities along and perpendicular to the common normal Hh, it is evident that the component along the normal must be that which transmits motion from the driver to the follower; for the component perpendicular to the normal, acting tangentially to the two curves, can evidently transmit no motion from the one to the other.

In considering the communication of motion through a very small angle, we may substitute for the curves, circular arcs having the same curvature as the curves themselves at the point of contact. The centres of these circular arcs will evidently lie at some points P and Q in the normal Hh. But the length PQ, being thus equal to the sum of the radii of two circles, will be constant during the small motion considered. Hence, joining AP and BQ, we find that the angular motion of the two contact curves, for that instant, will be exactly the same as that of two arms, AP and BQ, connected by the link PQ. Hence, the relation of the angular velocities will be expressed in the same manner as in Art. 24.

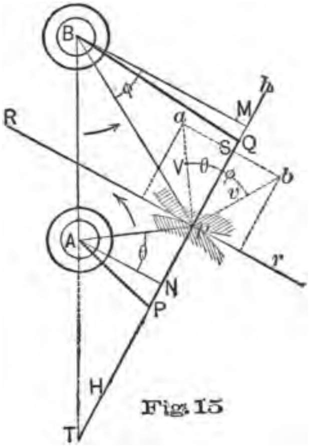


Fig. 15

Letting a = angular velocity of lower piece about A, and
a' = angular velocity of upper piece about B,

we have, as before,

$$\frac{a'}{a} = \frac{AN}{BM} = \frac{AT}{BT}.$$

Hence, in the communication of motion by contact, —

1. The angular velocities of the pieces are inversely proportional to the perpendiculars from the fixed centres of motion upon the common normal.

2. The angular velocities of the pieces are inversely proportional to the segments into which the common normal divides the line of centres.

30. Directional Relation. — From Figs. 14 and 15, it is evident that the directional relation of the rotations depends on the position of the centres A and B with reference to the normal Hh. If they are on the same side of Hh, the rotations will take place in the same direction; if on opposite sides, the rotations will take place in contrary directions.

31. Condition of Constant Velocity Ratio. — The value of the velocity ratio (Art. 29) is

$$\frac{a'}{a} = \frac{AT}{BT}.$$

Now, in order that this expression shall have a *constant* value, the ratio of AT to BT must remain unchanged. But, as $AT + BT = AB$, which is itself constant, it follows, that, in order to preserve the constancy of the above ratio, the actual lengths of AT and BT must not vary; in other words, the point T must remain fixed in position. Hence we see, that, in order to obtain a constant velocity ratio in contact motions, the curves must be such that their common normal at the point of contact shall always cut the line of centres at the same point.

32. Condition of Rolling Contact. — It has been stated (Art. 29) that the components of the velocities V and v, Figs. 14 and 15, along the common normal Hh, represent the transmitted motion. As pointed out by Professor Rankine, these components must be equal; or $V \cos\theta = v \cos\phi$, using the same notation as in Art. 24. If this equation were not true, either one curve would move away from the other, or else one would intersect the other, both of which are manifestly impossible while contact exists.

As the components of V and v along the tangent Rr must represent the lost motion, it is evident that the velocity with which the curves are sliding over each other will be represented by the difference between these tangential components in the case of Fig. 14, and by their sum in the case of Fig. 15. In other words, the velocity of sliding is $V \sin\theta - v \sin\phi$ in the former case, and $V \sin\theta + v \sin\phi$ in the latter. In order that there shall be pure rolling contact, it is evidently necessary that these expressions shall be equal to zero.

Thus, we must have in Fig. 14,

$$V \sin\theta - v \sin\phi = 0.$$

$$\therefore \quad V \sin\theta = v \sin\phi. \qquad (1)$$

But, as above stated,

$$V \cos\theta = v \cos\phi. \qquad (2)$$

Dividing (1) by (2) we get

$$\tan\theta = \tan\phi, \qquad (3)$$

which can only be true when $\theta = \phi$; in other words, when V coincides in direction with v. But V and v are perpendicular to Ap and Bp respectively; hence the latter must also coincide in direction, and as A and B are fixed, it follows that p must lie in the line of centres AB. The *condition of rolling contact*, then, for curves revolving in the same plane about parallel axes, is that the point of contact shall always lie in the line of centres.

In most cases of sliding contact, the point of contact is not fixed in position, and the amount of sliding will vary with the distance of the point of contact from the line of centres. We have found (Art. 29) that the velocity ratio in contact motions depends on the position of the point of intersection of the common normal with the line of centres. But in rolling contact, the point of contact must lie on the line of centres, and must hence be identical with the point of intersection just mentioned. Hence, in rolling contact, the angular velocities of the pieces are inversely proportional to the segments into which the point of contact divides the line of centres.

33. Similarity in all Modes of Transmission. — By comparing the deductions of Arts. 24–30, we find a great similarity between the various modes of transmitting motion, so far as the velocity ratio and the directional relation in the various cases are concerned.

If we designate by *line of action* the line of the link in link-work, the line of the wrapping connector, and the common normal in contact motions, we may express the laws governing the action of any elementary combination in which the pieces rotate about fixed parallel axes as follows:—

1. The angular velocities are inversely proportional to the perpendiculars let fall from the centres of motion upon the line of action.

2. The angular velocities are inversely proportional to the segments into which the line of action divides the line of centres.

3. The rotations have the same direction if the centres of motion lie on the same side of the line of action, and contrary directions if they lie on opposite sides of that line.

CHAPTER III.

COMMUNICATION OF MOTION BY ROLLING CONTACT.
VELOCITY RATIO CONSTANT.
DIRECTIONAL RELATION CONSTANT.

Cylinders. — Cones. — Hyperboloids. — Practical Applications. — Classification of Gearing.

34. It has been shown (Art. 32) that, in the rolling contact of curved pieces revolving in the same plane about fixed parallel axes, the point of contact will always lie in the line of centres, and that the angular velocities are inversely proportional to the segments into which the point of contact divides that line.

Therefore, if the velocitiy ratio of two such pieces in rolling contact is constant, these segments must be constant, and the curves must have a constant radius; in other words, the curves must be circular arcs turning about their centres, and no other curves will satisfy the conditions.

Axes Parallel.

35. Rolling Cylinders.—In Fig. 16, let AC, BD, be parallel axes mounted in a framework, by which they are kept at a constant distance from each other. Let E and F be two cylinders, fixed opposite to each other, one on each axis, and concentric with it; the sum of their radii being equal to the distance between the axes.

The cylinders will, therefore, be in contact in all positions, the line of contact being a common element of both. If one cylinder be made to rotate, it will drive the other by rolling contact, and compel it to rotate. The linear velocity of every point in the cylindrical surface of either wheel must evidently be the same.

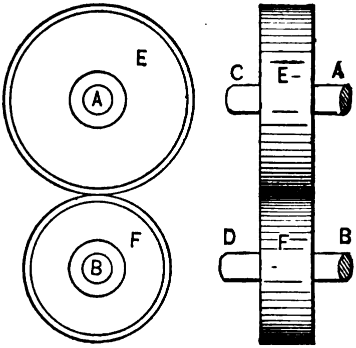


Fig. 16

Let R be the radius of the driver, and r the radius of the follower. Let the circumference of the driver be divided into N equal parts, and let the circumference of the follower contain n of these parts. Let P and p be the periods or times of rotation; L and l the number of rotations in a given time, or the *synchronal rotations* of driver and follower respectively; and, as before, let a and a' be their angular velocities. Then

$$\frac{a'}{a} = \frac{R}{r} = \frac{N}{n} = \frac{P}{p} = \frac{l}{L};$$

and it is evident that these ratios will hold, without regard to the *absolute* velocities.

36. If the cylinders roll together by external contact, as in Fig. 16, they will evidently rotate in *opposite* directions.

If it is desired to have them rotate in the *same* direction, one wheel is given the form of an *annulus*, or ring, as in Fig. 17, to which the other wheel is tangent internally. The rolling surfaces are cylinders, as before; the line of contact is an element of both cylinders; and the relations stated in the last article are equally true for this case, the only change being, that the rotations now take place in the same direction. The difference of the radii is evidently the distances between centres. Thus, if we have given the distance between two axes, and the velocity ratio of driver and follower, expressed in any of the above terms, we can readily find the radii of wheels which will answer the given conditions.

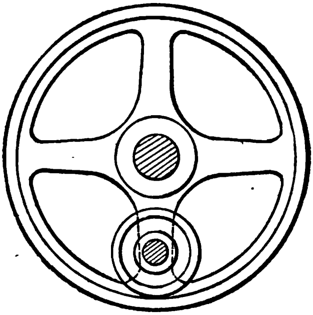


Fig. 17

If the axes of rotation are not parallel, they may or may not intersect; and these cases will be considered separately.

Axes Intersecting.

37. Rolling Cones. — The conclusions arrived at in Art. 34 follow directly from our propositions concerning rolling contact; the circles in contact being in the same plane, and rotating about fixed parallel axes. A little consideration will, however, make it clear, that, if the axes be turned in their common plane about the point of contact of the two circles, the latter will, at any angle, have a common tangent at this point. This tangent will be the line of intersection of the planes in which the two circles lie. Both circles will be in true rolling contact with this common tangent, and hence with each other; and their perimetral and angular velocities will be the same as before.

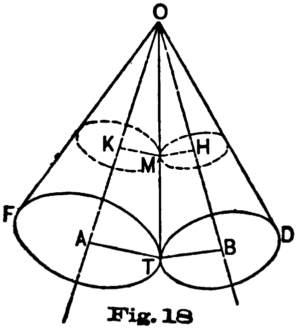


Fig. 18

In Fig. 18, let *OA*, *OB*, be two axes which intersect at *O*; and let the two right cones *OTD*, *OTF*, be constructed on these axes, the cones having a common element *OT*. If through any point *M* in *OT* we pass planes perpendicular to the axes *OA* and *OB*, the sections of the cones will be circles which will be in contact at *M*; and a constant velocity ratio will be maintained between the axes by means of these circles. For the angular velocities of these circles are, as before,

$$\frac{\alpha'}{\alpha} = \frac{MK}{MH} = \frac{AT}{BT},$$

a constant ratio; therefore the two cones will rotate in true rolling contact, and their angular velocities will be inversely proportional to the perpendiculars from any point of the common element on the axes. The relations of angular velocities, periods, etc., will evidently be the same as for two cylinders whose radii are in the same proportion as the radii of the bases of the cones.

38. Having given the positions of the axes, and the velocity ratio, it is required to construct the cones.

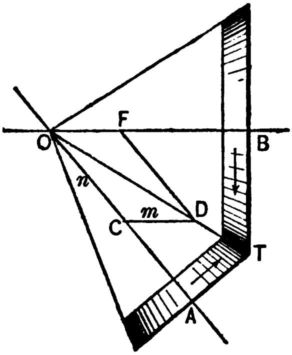


Fig. 19

In Fig. 19, let OA be the driving axis, and OB the following axis: and let the velocity ratio of driver to follower be $\frac{a'}{a} = \frac{m}{n}$; in other words, OA is to make n revolutions while OB makes m revolutions. On OA lay off OC equal to n divisions on any convenient scale. Through C draw CD parallel to OB, and make it equal in length to m divisions of the same scale. Through D draw ODT, which will be the line of contact. From any point T of ODT, let fall the perpendiculars AT and BT on the axes. If we now construct two right cones on these axes, having AT and BT as radii of their respective bases, these cones will roll together with the required velocity ratio; for, from the figure, we have

$$\frac{a'}{a} = \frac{m}{n} = \frac{\sin COD}{\sin ODC} = \frac{\sin COD}{\sin BOD} = \frac{AT}{OT} \div \frac{BT}{OT} = \frac{AT}{BT}.$$

In other words, the radii of the bases have the required relation.

39. It is usual in practice to employ, not the whole cones, but only thin frusta of them, as shown in Figs. 19 to 24.

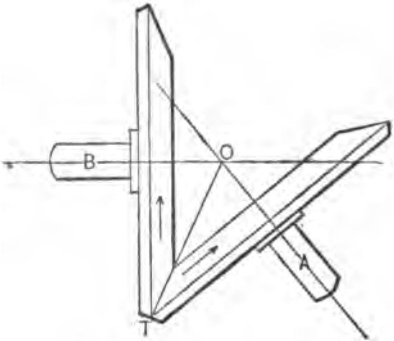


Fig. 20

In Fig. 19, the common element is located in the acute angle between the intersecting axes; but it may as readily be placed in the obtuse angle, the location depending on the exact data of the problem. Examples of different arrangements are shown in Figs. 20, 21, and 22. In these figures the angles of intersection of the axes are the same as in Fig. 19,

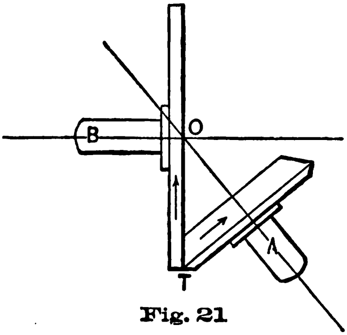


Fig. 21

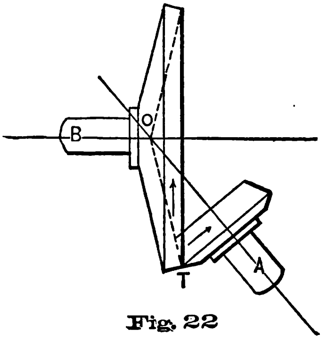


Fig. 22

but the velocity ratio and the directional relation may be varied at pleasure. In Figs. 19 and 20 the velocity ratio is different, and the direction of rotation of the follower is also changed in the latter by moving the element of contact from the acute to the obtuse angle. In Figs. 21 and 22 the directional relation is the same as that of Fig. 20; but, by altering the velocity ratio, one of the cones becomes a *flat* disc in one case, and a *concave* conical surface in the other.

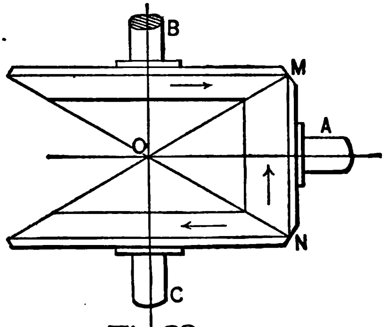


Fig. 23

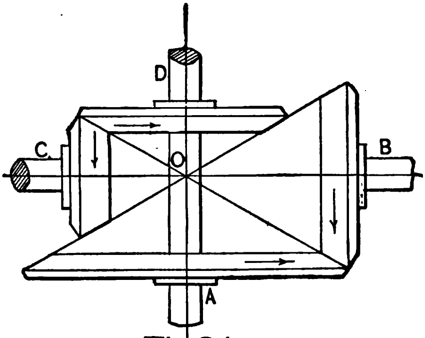


Fig. 24

40. Thus far we have considered only those cases in which the axes intersect obliquely; but in practice the axes intersect most frequently at right angles, as in Fig. 23. In this case it will be noticed that the cones are in contact along two elements, *OM* and *ON*, and that the followers will rotate in opposite directions. Thus, in Fig. 23, where *A* is the driver, the two followers, *B* and *C*, rotate in opposite directions, as shown by the arrows. But if, as in Fig. 24, the driving axis *A* be continued beyond the common vertex of the cones, and two other frusta be constructed, motion will be given to the two followers *B* and *C* in the same direction; the velocity ratio of both pairs of frusta being the same.

Axes neither Parallel nor Intersecting.

41. Hyperboloid of Revolution.—When the axes do not lie in the same plane, motion may be transmitted from the one to the other by means of surfaces, known as hyperboloids of revolution. The hyperboloid of revolution is the warped surface generated by a right line revolving about another right line not in the same plane with the first. Its form and the manner of constructing it are shown in Fig. 25, both vertical and horizontal projections being employed for the sake of clearness. Let the axis be taken vertical; it will be horizontally projected at *O′* and vertically at *Cc*. The revolving line, or *generatrix*, is, for convenience, taken in a position parallel to the vertical plane of projection, and is shown at *MN*, *M′N′*. As this line revolves about the axis, any point, *P*, *P*, of the line describes a circle, whose radius is projected vertically at *OP*, and horizontally *in its true length* at *O′P′*. Draw the common perpendicular to the two lines *MN* and *Cc*. It will be projected horizontally *in its true length* at *O′D′*, and vertically in the point *D*. The circles described by different points of the line *MN* will evidently vary in size; the largest being described by the points *M* and

N respectively, and the smallest by the point *D*. To construct the projections of the curved surface, we must find the

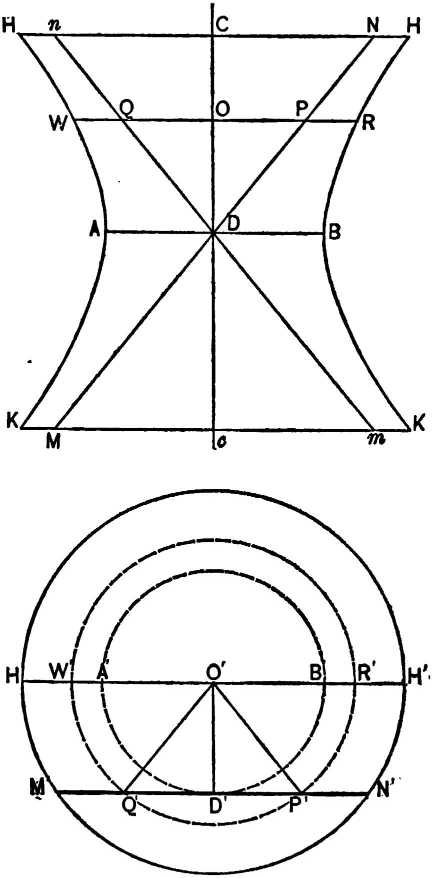


Fig. 25

projections of the circle described by any point *P* of the line *MN*. Its horizontal projection will be the circle *W'P'R'*; while its vertical projection will be the straight line *WPR*, and R and W will be points of the meridian curve. By repeating this process for a sufficient number of points of the line MN, the meridian curve may be drawn; and it will be found to be a *hyperbola*. The circle $A'D'B'$, described by the point D' (which is the intersection of the generatrix with the common perpendicular $O'D'$), is called the *circle of the gorge;* and the circles described by the points M' and N' are called the circles of the lower and upper bases respectively.

If we take the line mn, parallel to the vertical plane of projection, intersecting MN at D, and making angle $nmc =$ angle NMc, and revolve it about Cc, we will evidently generate the same surface as before; for the paths of m and M coincide, as do also those of n and N, and the point D is common to both lines: hence any two points, one on each line, equidistant from D, such as P and Q, will describe the same circle.

Through any point of the surface, then, two rectilinear elements, or *generatrices*, may be drawn; and their projections on a plane perpendicular to the axis will be tangent to the projection of the *gorge circle* on that plane.

42. Rolling Hyperboloids.—If through any point of a surface two lines of the surface be drawn, the plane which contains the tangents to both these lines will be tangent to the surface at that point. Hence, if through any point of the curved surface of a hyperboloid we pass two intersecting generatrices, the plane containing these two elements will be tangent to the surface at that point. The normal to the surface at that point must, of course, be perpendicular to that tangent plane; and, as the surface is one of revolution, it must intersect the axis.

If a series of such normals be drawn through different points of the revolving line, they will lie in planes perpendicular to the latter, and therefore parallel to each other. Suppose three planes to be drawn parallel to both the axis Cc and the generatrix MN; one through the axis, another through the generatrix, and the third at any convenient distance. Conceive a number of points to be laid off at definite and equal intervals on the line *MN*. Now, in passing along *MN* from one point to the other, the normal, though always remaining perpendicular to *MN*, will still turn about the latter, so that its other end will describe on the plane through the axis a straight line; viz., the axis itself. Now, as these three planes are parallel, and the normal moves so that its two ends trace straight lines on two of the planes, it is evident that the prolongation of the normal will trace a straight line on the third plane. This straight line may be taken as a new axis; and by revolving *MN*, the generatrix of the first hyperboloid, about this new axis, a second hyperboloid will be generated: and these two surfaces will, by construction, have a common normal at every point of the element of contact *MN*, and will be tangent to each other all along that element. If one of these hyperboloids be now rotated about its axis, it will drive the other by a mixture of rolling and sliding contact; the sliding taking place in the direction of the element of contact, and the rolling in a direction perpendicular to that element.

43. Velocity Ratio of Rolling Hyperboloids. — In Fig. 26 we have two hyperboloids in contact along the line *MN*, and revolving about the axes *Oo* and *Rr* respectively. Let *P* be the point of contact of the gorge circles *APL* and *BPK*, and let the inclined hyperboloid be the driver. Then at any instant, any point on the gorge circle of either hyperboloid will be moving in the direction of the tangent to that circle; and the vertical projection of the tangent must evidently be perpendicular to that of the axis of the hyperboloid. Draw *Pa* perpendicular to *Rr*, and of proper length to represent the velocity *V* of the point *P* of the driver at any instant. Similarly, draw *Pb* to represent the velocity *v* of the point *P* of the follower at the same instant.

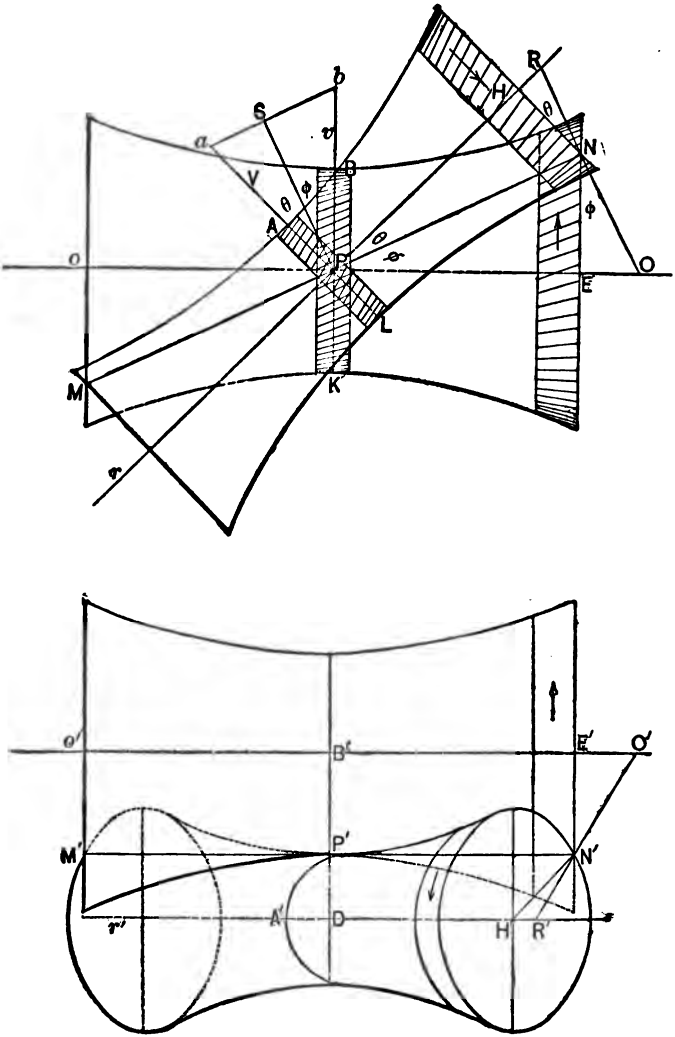


Fig. 26

Let a = angular velocity of driver, and
a' = angular velocity of follower.

$$a = \frac{V}{PA} = \frac{V}{P'D},$$

$$a' = \frac{v}{PB} = \frac{v}{P'B'},$$

$$\frac{a'}{a} = \frac{v}{V} \times \frac{P'D}{P'B'}. \qquad (1)$$

Draw the common normal to both surfaces at the point N. As the line MN is parallel to the vertical plane of projection, its vertical projection will evidently be perpendicular to that of the normal. Hence, for the vertical projection of the normal, draw ONR, perpendicular to MN at N. O and R being points on the axes, we readily find its horizontal projection $O'N'R'$. As O, N, and R are respectively the vertical projections of three points on a straight line whose horizontal projections are O', N', and R', we have the ratio

$$\frac{N'R'}{O'N'} = \frac{NR}{ON}.$$

But $NR = PN \tan\theta$, and $ON = PN \tan\phi$,
hence

$$\frac{N'R'}{O'N'} = \frac{\tan\theta}{\tan\phi}. \qquad (2)$$

As $B'O'$, $P'N'$, and DR' are parallel straight lines, we have

$$\frac{P'D}{P'B'} = \frac{N'R'}{O'N'} = \frac{\tan\theta}{\tan\phi}. \qquad (3)$$

Substituting in (1) the value of $\frac{P'D}{P'B'}$, just found, we have

$$\frac{a'}{a} = \frac{v}{V} \times \frac{\tan\theta}{\tan\phi}. \qquad (4)$$

The actual transmission of motion between the hyperboloids takes place by pure rolling contact in a direction *perpendicular* to the common element passing through the points of contact; for the sliding contact which takes place *along* that element can, of course, transmit no motion. Drawing PS perpendicular to MN, and noting that in rolling contact, each pair of points of the surfaces which are in contact at a given instant, must at that instant be moving in the same direction with the same velocity, we have

$$V \cos\theta = v \cos\phi, \qquad (5)$$

$$\frac{v}{V} = \frac{\cos\theta}{\cos\phi}. \qquad (6)$$

Substituting in (4) the value of $\frac{v}{V}$, just found, we have

$$\frac{\alpha'}{\alpha} = \frac{\cos\theta \tan\theta}{\cos\phi \tan\phi} = \frac{\sin\theta}{\sin\phi}. \qquad (7)$$

But $\sin\theta = \frac{NH}{PN}$ and $\sin\phi = \frac{EN}{PN}$, hence

$$\frac{\alpha'}{\alpha} = \frac{NH}{EN}. \qquad (8)$$

Hence, projecting both axes and the common element on a plane parallel to all of them, as in Fig. 26, we find that the angular velocities of the hyperboloids are inversely proportional to the sines of the angles made by the projection of the common element with the projections of the respective axes; that is, inversely proportional to the projections of the perpendiculars let fall on the axes from any point of the common element.

The radii of the gorge circles are directly proportional to the tangents of these angles; that is, to the projections of the lines drawn from any point of the common element perpendicular to the latter, and terminating in the axes.

44. While the components of V and v along the line of contact must evidently have a *constant* value for all points of the same hyperboloid, yet as the angular velocities of all points of the hyperboloid must be the same, it follows that the values of V and v themselves must *increase* as the points considered are farther from the gorge circle, and hence that the *percentage* of sliding must *decrease* at the same rate.

45. Having given the positions of the axes, and the velocity ratio, it is required to construct the hyperboloids.

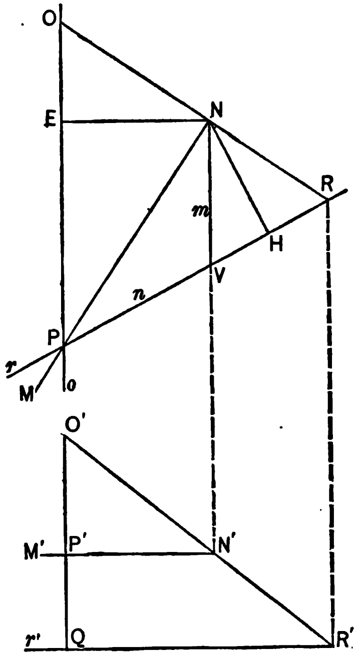


Fig. 27

In Fig. 27, let Rr, $R'r'$, and Oo, O', be the projections of the driving and following axes respectively; the vertical plane of projection being taken parallel to both axes. Let the driver be required to make n revolutions while the follower makes m revolutions; in other words, $\frac{a'}{a} = \frac{m}{n}$.

On PR lay off PV equal to n divisions on any convenient scale. Through V draw VN parallel to Oo, and equal to m divisions of the same scale. From N let fall EN and NH perpendicular to Oo and Rr respectively. Through N and P draw the line NP. This will be the vertical projection of the element of contact; for, from the triangles in the figure,

$$\frac{m}{n} = \frac{NV}{PV} = \frac{\sin NPV}{\sin PNV} = \frac{\sin NPV}{\sin EPN} = \frac{NH}{EN} = \frac{a'}{a}.$$

Through N draw ONR perpendicular to NP. ONR is the vertical projection of the normal; hence the horizontal projection of the point R must be at R' on the horizontal projection, $R'r'$, of the axis Rr. Joining $O'R'$, we have the horizontal projection of the normal. Projecting N horizontally at N' on $O'R'$, and drawing $N'P'$ parallel to $R'r'$, we have the horizontal projections of the common element and the gorge circle radii $O'P'$ and $P'Q$. We have thus determined all the data necessary to the construction of the hyperboloids, as explained in Art. 41.

As in the case of cones, only thin frusta of these hyperboloids (Fig. 26) are used in practice; and their location is optional, except that, as already indicated, the percentage of sliding increases as they come nearer the gorge planes.

46. Analogy between Cones and Hyperboloids.— As the radii of the gorge circles are made smaller, the meridian curves of the hyperboloids will become flatter, and the surfaces will begin to approximate to the conical shape. When the radii of both gorge circles reduce to zero, the axes will intersect, and the hyperboloids will become true cones; the element of contact lying in the plane of the axes, and passing through their point of intersection. Cones, then, may be considered as the limiting case of hyperboloids; and it will be found, that, under similar conditions, they will present similar peculiarities of arrangement. From the similarity of the solutions in Arts. 38 and 45, it is obvious that we may use our discretion in locating the common element in the case of the hyperboloids, just as explained in Art. 39 for the case of cones. By changing the common element from the acute angle between the projections of the axes, to the obtuse angle (a change similar to that shown by Figs. 19 and 20), we will change the directional relation of the hyperboloids. Again: by varying the velocity ratio so as to divide the angle in the same ratio as in Figs. 21 and 22, we will reduce one hyperboloid to a *flat* disc in one case, and to a *hollow* hyperbolic surface in the other.

47. The case of axes neither parallel nor intersecting may also be solved by means of two pairs of cones.

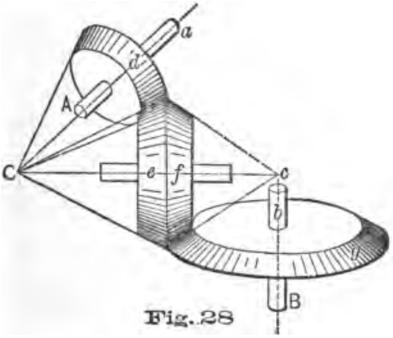


Fig. 28

In Fig. 28, let *Aa*, *Bb*, be the driving and following axes respectively. Draw the line *Cc* intersecting the two axes in the points *C* and *c*, and let an intermediate axis be taken in this line. Now, a pair of rolling cones, *d* and *e*, having their common apex at *C*, will communicate motion from the axis *Aa* to the intermediate axis *Cc*; and a pair of rolling cones, f and g, having their common apex at c, will transmit motion from the intermediate axis Cc to the axis Bb. By this means the rotation of Aa is transmitted, by pure rolling contact, to Bb.

Let a, a', and a'' be the angular velocities of the axes Aa, Bb, and Cc respectively, and R, r, and R' the radii of the bases of their cones, those of the cones e and f being the same. Then

$$\frac{a''}{a} = \frac{R}{R'}, \text{ also } \frac{a'}{a''} = \frac{R'}{r}; \text{ hence } \frac{a'}{a} = \frac{R}{r};$$

exactly as if the cones d and g were in immediate contact.

Practical Applications.

48. We have now determined the theoretical forms required to transmit motion by rolling contact with a constant velocity ratio, but the successful application of these forms in practice requires certain changes or substitutions to be made. It is impossible to transmit motion against any considerable resistance by means of such smooth surfaces, and hence various expedients are resorted to in order to obtain the necessary adhesion.

49. Friction Gearing. — For light machinery, and in cases where a constant velocity ratio is not imperative, the rolling pieces may be made of different materials; for instance, one may be made of wood and the other of iron. In this case, the iron wheel should be the follower. Again: one of the wheels may be covered with leather, or rubber, or other elastic material. To secure the necessary amount of adhesion in such cases, the rotating pieces are kept in contact and pressed together by adjusting their bearings, or applying weights or springs.

50. Grooved Friction Gearing. — Another method is shown in Fig. 29. The wheels are provided with angular grooves, shown in an enlarged section on the left. The angle between *ab* and *cd* is usually about forty to fifty degrees. The adhesion is greatly increased by this means, and is obtained, as before, by pressing the wheels together. Such wheels are widely used for hoisting-engines, and are generally made of cast-iron.

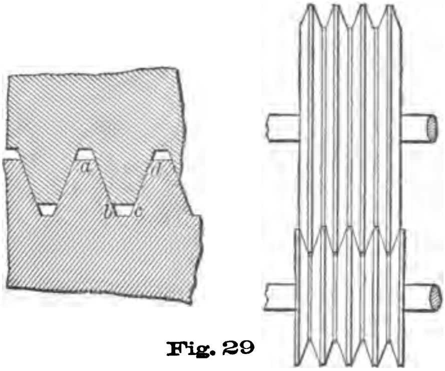


Fig. 29

51. Gearing. —The method in most general use for the prevention of slipping between rotating pieces is, to form *teeth* upon them.

Gearing is the general term which includes all forms of mechanistic devices in which the motion is transmitted by means of teeth. The contact surface of the rotating pieces is called the *pitch surface*, and its intersection with a plane perpendicular to the axis of rotation is termed the *pitch line.* This line is the basis of all calculations for velocity ratios and for the construction of teeth. The *pitch line* in the cases in which the velocity ratio is constant evidently becomes a *pitch circle.*

52. Classification of Gearing. — Gearing is divided into classes according to the form of the pitch surfaces for which the toothed wheels are the equivalents. There are five such classes; namely, *spur* gearing, *bevel* gearing, *skew* gearing, *screw* gearing, and *face* gearing.

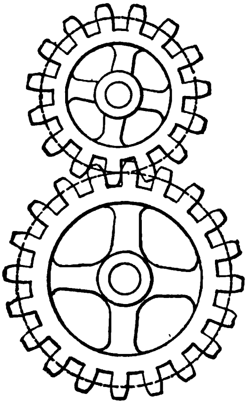


Fig. 30

53. In Spur Gearing, illustrated by Fig. 30, the pitch surfaces are cylinders, and the teeth engage along straight lines which are parallel to the elements of the cylinders. A spur wheel having a small number of teeth is usually called

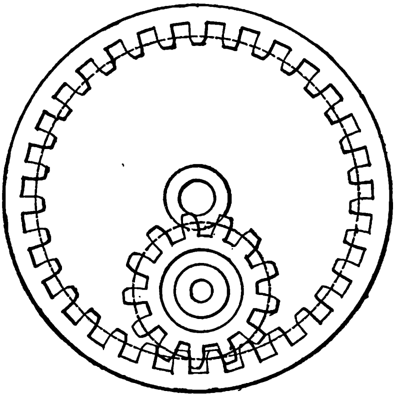


Fig. 31

a *pinion.* When the teeth are formed on the inside of a ring, as shown in Fig. 31, the wheel is termed an *annular wheel.* In this case, as before pointed out, the directions of rotation of driver and follower are the same; while in the case of two spur wheels, the directions are opposite to each other. As the diameter of the pitch circle of a wheel increases, its curvature becomes less and less, and finally disappears when the former becomes infinite. In this case the toothed piece is called a *rack* (Fig. 32), and its pitch line is the straight line tangent to the pitch circle of the wheel with which it works. In Figs. 30, 31, and 32, the various pitch lines are shown dotted.

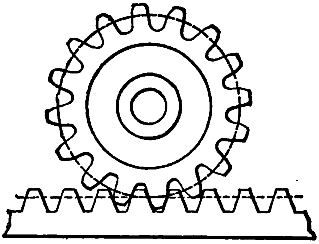


Fig. 32

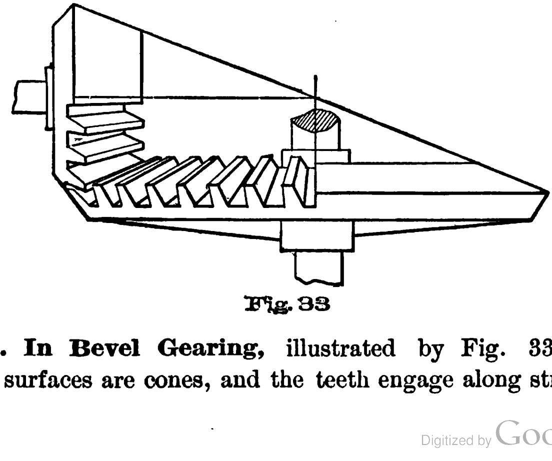


Fig. 33

54. In Bevel Gearing, illustrated by Fig. 33, the pitch surfaces are cones, and the teeth engage along straight lines the directions of which must all pass through the common vertex of the two cones. In actual wheels, the teeth are, of course, placed all around the frusta; but in the figure they are drawn only on part of the wheels, in order to show more clearly the relation in which they stand to the pitch surfaces. When the axes are at right angles, and two bevel wheels are constructed on equal cones, the line of contact making an angle of forty-five degrees with each axis, or, in other words, the velocity ratio being unity, the wheels are termed *mitre gears.*

55. In Skew Gearing, illustrated by Fig. 34, the pitch surfaces are hyperboloids of revolution. The teeth of these wheels engage in lines which approximate, in their general direction, to that of the common element of the two hyperboloids. This class of gearing is not often used, owing to the difficulty of forming the teeth; the usual method for axes neither parallel nor intersecting being, to employ the intermediate cones described in Art. 47.

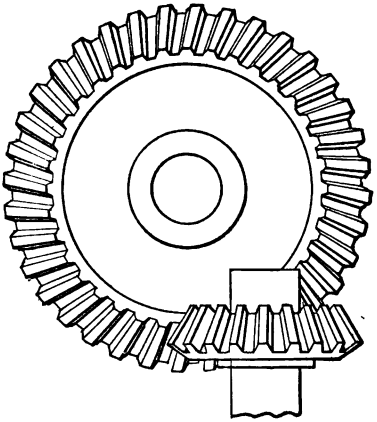


Fig. 34

56. In Screw Gearing, illustrated by Fig. 35, the pitch surfaces are cylinders whose axes are neither parallel nor intersecting; and hence the cylinders touch each other at *one point only*. The lines upon which the teeth are constructed are helices on the surfaces of these cylinders. Motion is transmitted by a purely helical or screw-like motion.

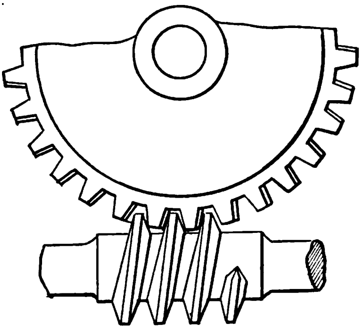


Fig. 35

57. In Face Gearing, illustrated by Fig. 36, the teeth are pins usually arranged in a circle, and secured to a flat circular disc fixed on the axis. Thus the contact is only between *points* of the surfaces of the pins. In Fig. 36

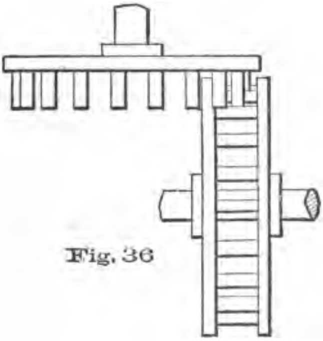


Fig. 36

the wheels are in planes perpendicular to each other, and the perpendicular distance between the axes is equal to the diameter of the pins, which in this case are cylindrical. This class of gearing is best adapted to wooden mill machinery, and has been used for that purpose almost exclusively.

58. Twisted Gearing. — In Fig. 38 is illustrated another form of gearing, sometimes called *twisted* gearing. It may be regarded as obtained from the *stepped wheel* shown in Fig. 37. The latter may be produced by cutting an ordinary spur wheel by several planes perpendicular to the axis, turning each portion through a small angle, and then securing them all together. By placing this wheel in gear with another, made in a similar manner, we combine the advantage of strength of large teeth with the smoothness of action of small ones. If the number of cutting planes be indefinitely increased, and each section be turned through an exceedingly small angle, it is clear that a *twisted* wheel, such as shown in Fig. 38, will be the result. But instead of ordinary spur teeth, whose elements are parallel to the axis of the wheel, we now have teeth whose elements have the directions of helices. The result is, that, in addition

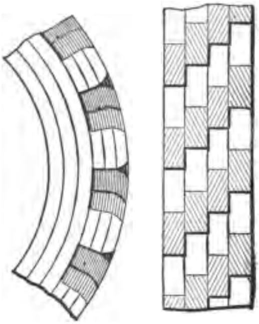


Fig. 37

to the pressure producing the rotation, there will be a pressure produced in the direction of the axis, tending to slide the wheels out of gear.

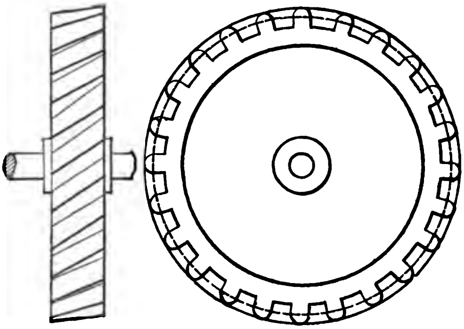


Fig. 38

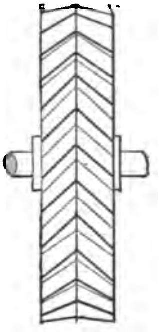


Fig. 39

The endlong pressure on the bearings may be prevented by the use of a wheel such as is shown in Fig. 39. By this arrangement, there is no longitudinal pressure on the bearings whatever, and the wheels run in gear with a smoothness of action unsurpassed by any other kind of gearing.

CHAPTER IV.

COMMUNICATION OF MOTION BY ROLLING CONTACT.

VELOCITY RATIO VARYING.

DIRECTIONAL RELATION CONSTANT.

Logarithmic Spirals.—Ellipses.—Lobed Wheels.—Intermittent Motion.—Mangle Wheels.

59. It has been shown (Art. 32) that, in the rolling contact of curves revolving in the same plane about fixed parallel axes, the point of contact always lies in the line of centres. The radii of contact coincide with this line; and at the point of contact the curves have a common tangent which must make equal angles, on opposite sides of the line of centres, with the two radii of contact.

60. In the preceding chapter, the ratio of the radii of contact was constant, and hence the velocity ratio was constant. If the curves are such that the radii of contact vary, the point of contact moving along the line of centres, the velocity ratio must vary. The *sum* of the lengths of each pair of the radii of contact must evidently be constant if the point of contact lies between the axes, or their *difference* must be constant if the axes lie on the same side of the point of contact.

61. The Logarithmic Spiral is a curve having the property, that the tangent makes a constant angle with the radius vector. Let two equal logarithmic spirals be placed reverse positions, and turned about their respective poles as fixed centres until the curves are in contact. Each of the radii of contact is a radius vector of the curve in which it lies, and hence both radii make the same angle with the common tangent at the point of contact. But this can only be true if the radii of contact lie in one straight line, namely, the line of centres; in other words, the point of contact lies on the line of centres, and equal logarithmic spirals are therefore rolling curves.

62. To Construct the Logarithmic Spiral. — In Fig. 40, let *O* be the pole of the spiral, and let *A* and *B* be two points through which it is desired to draw the curve. From the property of the curve given above, namely, that the tangent makes a constant angle with the radius vector, it may readily be proved that, if a radius vector be drawn bisecting the angle between two other radii vectors, the former will be a mean proportional between the two latter. Draw the radii vectors *AO* and *BO*, and the line *OD* bisecting the angle *AOB*. Then, if *D* is a point of the curve, *OD* must be a mean proportional between *OA* and *OB*; in other words, $\frac{OA}{OD} = \frac{OD}{OB}$. On the straight line *AO* lay off $OC = OB$. On *AOC* as a diameter, describe the semi-circle *AEC*. Draw *OP* perpendicular to *AOC*. Then *OE* is a mean proportional

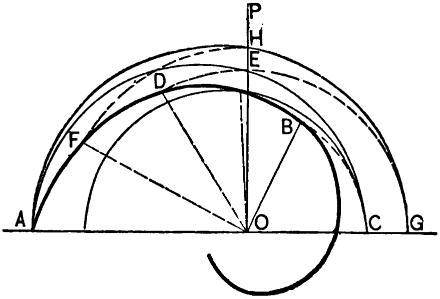


Fig. 40

between OA and OB. Therefore make $OD = OE$, and D will then be a point on the curve.

In the same manner, bisect the angle AOD, make OF a mean proportional between OA and OD to find the point F, and so on.

63. Since the logarithmic spiral is not a closed curve, two such spirals cannot be used for the transmission of continuous rotation; but they are well adapted for reciprocating circular motion.

In Fig. 41, let the distance between the axes A and B be given; and let it be required, that, while the driving axis A turns through a given angle, the velocity ratio shall vary between given limits.

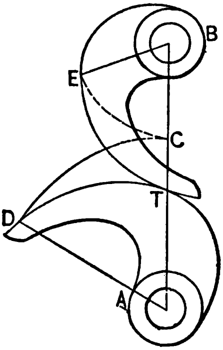


Fig. 41.

Divide AB at T into two segments whose ratio is one of the given limits, and at C into segments whose ratio is the other limit. Lay off the angle DAC equal to the given angle, and make $AD = AC$.

The problem is now simply to construct a logarithmic spiral (Art. 62) having the pole A, and passing through the points T and D.

The follower is necessarily a portion of the same curve in a reverse position; and the latter having been drawn about the pole B, draw arcs of circles about B with the radii BC and BT. The portion of the curve between the intersections of these arcs and the spiral will be the required edge of the follower.

Let α = angular velocity of driver, and α' = angular velocity of follower; then, while the driver turns from the position in the figure through the angle TAD, the velocity ratio will vary between the limits $\frac{\alpha'}{\alpha} = \frac{AT}{BT}$ and $\frac{\alpha'}{\alpha} = \frac{AC}{BC}$.

64. Rolling Ellipses.—In Fig. 42, let ETH and FTG be two similar and equal ellipses, placed in contact at a point T, such that the arcs ET and FT are equal; E and F being the extremities of the respective major axes. It is a property of the ellipse that the tangent CTD makes equal angles with the radii BT and bT, or AT and aT. Therefore the angle DTA = angle CTB, and angle DTb = angle CTa; hence BTA and bTa are straight lines. Also, since the arc ET = arc FT by construction, TA and Tb are equal; therefore $BT + TA = BT + Tb = FG = EH$, a constant length whatever the position of the point of contact, T. Similarly, $bT + Ta = FG = EH$. Hence two equal and

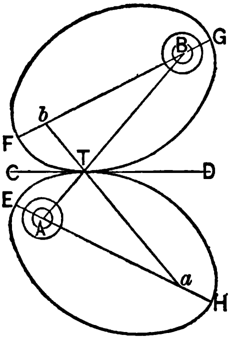


Fig. 42

similar ellipses can transmit motion between parallel axes by pure rolling contact; each ellipse turning about a focus as a fixed centre, and its major axis being equal to the distance between those centres. The velocity ratio will in this case vary between the limits $\frac{a'}{a} = \frac{AH}{BG} = \frac{AH}{AE}$ and $\frac{a'}{a} = \frac{AE}{FB} = \frac{AE}{AH}$, the two limits being reciprocals of each other.

65. Lobed Wheels. — By using rolling ellipses, as shown in the preceding article, we can obtain a varying velocity ratio having one maximum and one minimum value during each revolution.

But it may be necessary that there shall be two, three, or more maximum, alternating with as many minimum, values of the velocity ratio during each revolution.

Lobed wheels which will roll together and answer these conditions can be produced by several methods from the logarithmic spiral and the ellipse.

66. Lobed Wheels derived from the Logarithmic Spiral.—In Fig. 43, let A and B be two fixed parallel axes, and let it be required to communicate motion between them by wheels so constructed that the velocity ratio will have four maximum and four minimum values. Let $\frac{a'}{a} = \frac{BT}{AT}$ be one limit: then the other is necessarily the reciprocal of this, or $\frac{a'}{a} = \frac{AT}{BT}$.

Make the angles TAC and DBT equal to 45°. Make $BD = AT$ and $AC = BT$. Construct (Art. 65) the portion CT of a logarithmic spiral having A as the pole, and passing through the points C and T. Draw CF, TE, and TD, similar curves symmetrically placed with regard to BT and AC. We have thus constructed one lobe of each wheel; and, as the angles TAF and DBE each include one-fourth of a circumference, the quadrilobes can be completed as shown, and will roll together with the varying velocity ratio required.

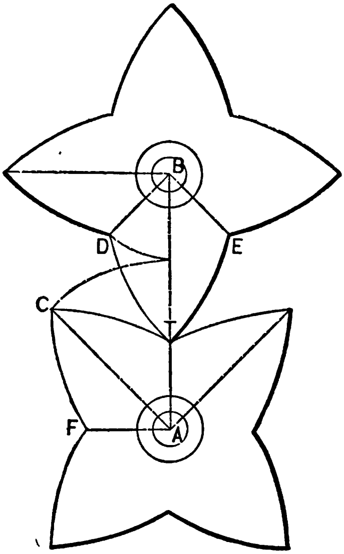


Fig. 43

The angles *TAF* and *DBE* may include any aliquot part of a circle; hence pairs of wheels with any desired number of lobes can be made in this way. They will roll together in similar pairs, unilobe with unilobe, bilobe with bilobe, and so on; but dissimilar pairs, such as one bilobe and one trilobe, will not roll together.

67. Lobed Wheels derived from the Ellipse.—Lobed wheels may be derived from rolling ellipses by the method of contracting angles, as illustrated by Fig. 44.

Let *A* and *B* be the fixed foci of two equal rolling ellipses in contact at *T*. Draw the radii *A*1, *A*2, etc., dividing the semi-ellipse *T*6 into equal angles about the focus *A*, and consequently into unequal arcs. If we describe arcs about *T* through the points 1, 2, 3, etc., cutting the other semi-ellipse $T6'$ at the points $1'$, $2'$, $3'$, etc., it is evident that the arc $T1 = T1'$, $T2 = T2'$, $T3 = T3'$, etc. Therefore the points 1 and $1'$, 2 and $2'$, etc., will come in contact on the line of centres AB; and $AB = A1 + B1' = A2 + B2' =$, etc. Bisect the angle $TA1$ by the line AI, and bisect the angle $TB1'$ by the line BI'. Make $AI = A1$, $BI' = B1'$. It is evident that these points, I and I', will come in contact on the line of centres when they have turned through the angles TAI ($= \frac{1}{2}$ angle $TA1$) and TBI' ($= \frac{1}{2}$ angle $TB1'$) respectively.

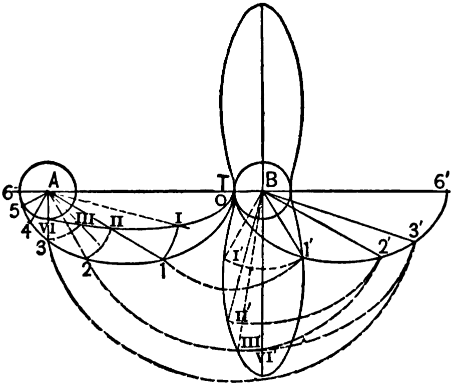


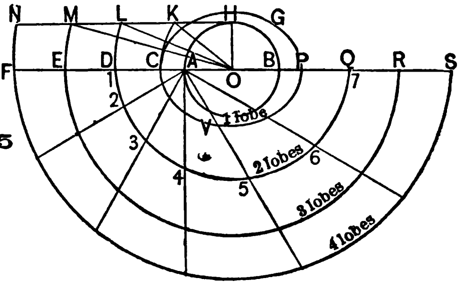
Fig. 44

Thus, if we find the series of points I, II, III, etc., and I', II', III', etc., in the manner just described, and draw through them two curves, as shown in the figure, they will be quadrants of two similar and equal bilobes, of which the remaining similar portions can then be readily drawn. From the above considerations, it is evident that these bilobes will roll together in perfect rolling contact. The velocity ratio will vary between $\frac{a'}{a} = \frac{AT}{BT}$ and $\frac{a'}{a} = \frac{BT}{AT}$. By contracting the angles to one-third, we can form the outlines of a pair of trilobes, and so on.

The wheels thus outlined will roll together in similar pairs, as bilobe with bilobe, trilobe with trilobe, and so on; but dissimilar pairs, such as one bilobe and one trilobe, will not roll together.

68. Interchangeable Lobed Wheels. — In Figs. 45 and 46 is illustrated a method of constructing lobed wheels from an ellipse, by which *any two wheels* of the set will roll together. The process of construction is simple and practical; but the rolling properties of the curves do not admit of simple demonstration, although they may readily be proved by graphical construction. In Fig. 45, let *A* and *B* be the foci of an ellipse, *CGPV*. Describe a circle about its centre *O* with a radius equal to the semi-focal distance *OA*. Draw the indefinite tangent *HN* parallel to *BA*. With radius *OC*, equal to the semi-major axis, and centre *O*, describe an arc *CK*, and lay off on the tangent the lengths *KL*, *LM*, and *MN*, equal to *HK*. From the centre *O* lay off on *OF* the distances *OC* = *OK*, *OD* = *OL*, *OE* = *OM*, and so on. With *OC*, *OD*, *OE*, etc., as semi-major axes, describe a series of concentric ellipses, having the common foci *A* and *B*. The primary ellipse is the curve required for the unilobe; the second ellipse, *DQ*, is the basis for the bilobe; the third, *ER*, for the trilobe; the fourth, *FS*, for the quadrilobe; and

Fig. 45



so on. Draw a semi-circle about *A*, and divide it into any number of equal angles by equidistant radii.

To form the bilobe (Fig. 46), divide a quadrant into the same number of equal angles as the semi-circle is divided, and on the equidistant radii in the quadrant lay off $B1' = A1$, $B2' = A2$, etc. Through the points 1′, 2′, 3′, etc., draw a curve: this will be one-fourth of the bilobe; the remaining portion of which, being symmetrical, can readily be drawn.

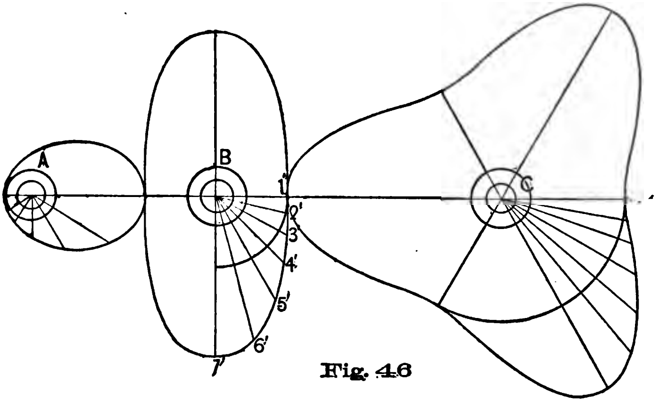


Fig. 46

For a trilobe, an angle of 60° is similarly divided, and the proper distances laid off on the equidistant radii in that angle. For a quadrilobe, we use an angle of 45°, and so on.

The velocity ratio of any two of these wheels in gear will vary between two limits, one of which will be the longest radius of the driver divided by the shortest radius of the follower, and the other the shortest radius of the driver divided by the longest radius of the follower.

69. Compulsory Rotation of Rolling Ellipses. — In the case of rolling ellipses (Fig. 42), it is evident that, when the motion takes place in the direction of the arrows, the radius of contact of the driver is *increasing* from *AE* to *AH*, and hence motion can be readily transmitted from the axis *A* to the axis *B*. But, when *H* has passed *G*, the radius of the driver is *decreasing*, and the driver will therefore tend to leave the follower. This can be prevented by forming teeth on the rolling faces of both pieces; but, if this is done, we no longer have pure rolling contact.

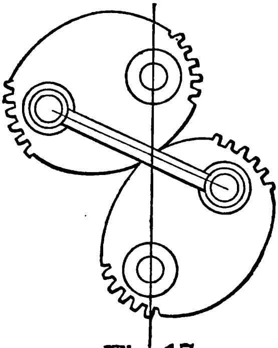


Fig. 47

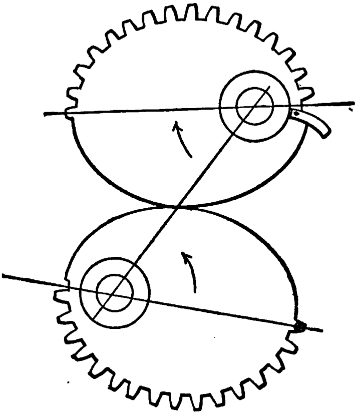


Fig. 48

When the position of the pieces will allow it, we can connect the free foci by means of a link, as in Fig. 47, since

(Art. 64) the distance between the free foci is constant in rolling ellipses. There will, however, be times during the revolution when the link will be in line with the fixed foci, and hence cannot transmit motion. This necessitates the formation of teeth on a small portion of each ellipse, near the ends of the major axis, as shown in Fig. 47.

Another method, when the revolution always takes place in the same direction, is, to form teeth on the *retreating* edge of the driver and the *corresponding* edge of the follower. In this case it is necessary to provide some means of insuring the proper contact of the teeth in order to prevent jamming. This may be done, as shown in Fig. 48, by placing a pin on the driver and a guide plate on the follower, which arrangement compels the first tooth to enter the proper space.

70. Intermittent Motion. — It may happen that the variation in the velocity ratio is to consist of an intermittent motion of the follower, while the driver revolves uniformly. In Fig. 49 is shown an intermittent motion formed from two spur wheels by cutting away the teeth of the driver on a

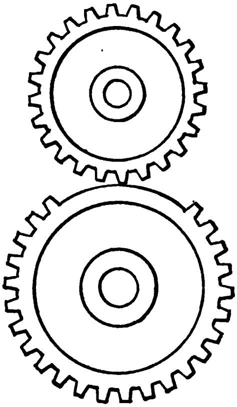


Fig. 49

portion of the circumference. There is the same objection to this method as before mentioned for elliptical wheels; namely, that the teeth are apt to jam after a period of rest of the follower. A partial remedy is the application of a pin and guide plate, similar to the arrangement shown in Fig. 48. A more complete motion is shown in Fig. 50. A portion of the driver is a plain disc of a radius greater than the pitch circle of the driver. A portion of the follower is cut away, to correspond to this; so that, while there is a slight clearance between the two faces, the follower is prevented from turning until the pin and curved piece come in contact.

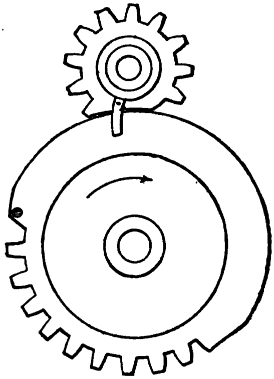


Fig. 50

Velocity Ratio Varying. Directional Relation Changing.

71. Mangle Wheels. — By combining a *spur* wheel with an *annular* wheel, we obtain a *mangle wheel*, as shown in Fig. 51. The direction of rotation is changed by causing the pinion, which always revolves uniformly in the same direction, to act alternately on the spur and on the annular portion.

The velocity ratio is constant during each partial revolution of the mangle wheel; but it is changed each time that the pinion passes from the spur to the annular portion, and vice versa. The pinion is mounted so that its shaft has a vibratory motion, working in a straight slot cut in the upright bar. The end of the pinion shaft is guided in the groove *CD*, the centre line of which is at a distance from the pitch lines of the mangle wheel equal to the pitch radius of the pinion. The pinion may also be mounted in a swinging frame, as indicated by dotted lines.

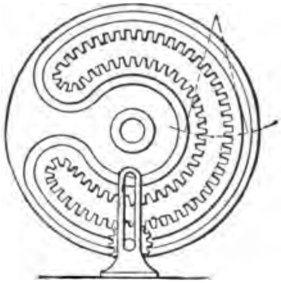


Fig. 51

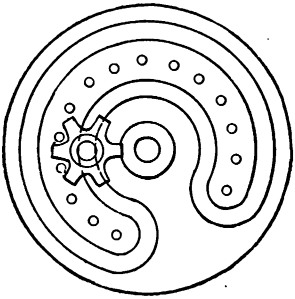


Fig. 52

If we construct the teeth of the spur and annular portions of the mangle wheel on the same pitch line, as in Fig. 52, we will obtain a combination in which the velocity ratio is *constant;* the directional relation changing, as in the preceding arrangement.

72. Mangle Rack.—A rack can be made in a similar manner to the above, and a reciprocating motion obtained from continuous rotation. Such motion is, however, more simply obtained by means of the pinion and double rack, shown in Fig. 53. Pins are placed on a portion of the face of the pinion, which engage with the pins of the rack above and below alternately, driving the rack back and forth.

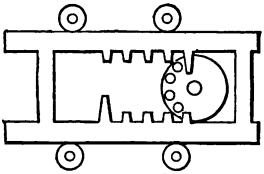


Fig. 53

CHAPTER V.

COMMUNICATION OF MOTION BY SLIDING CONTACT.

VELOCITY RATIO CONSTANT.

DIRECTIONAL RELATION CONSTANT.

TEETH OF WHEELS.

Special Curves. — Rectification of Circular Arcs. — Construction of Special Curves. — Circular Pitch. — Diametral Pitch.

73. General Problem. — It has been shown (Art. 32) that, in order to obtain a *constant* velocity ratio in contact motions, the axes of the pieces being parallel, the curves must be such that their common normal at the point of contact shall always cut the line of centres at the same point. The curved edge of one of the moving pieces may always be assumed at pleasure; the problem then being to find such a curve for the edge of the other, that, when motion is transmitted by the contact of these curved edges, the velocity ratio of the two axes may be constant. This problem is always capable of solution, theoretically at least; and, as the assumed curve may be of any shape whatever, we can obtain an infinite number of pairs of such curves. For practical purposes, there are certain definite curves in almost universal use, and these will be first discussed. In Chap. III. has been explained the method of finding the diameters of two pitch circles, which by their *rolling* contact shall transmit motion with a given velocity ratio. We now propose to show how to describe certain curves, which, when substituted for the circles, and caused to move each other by *sliding* contact, shall exactly replace the *rolling* action of the circles, so far as relates to the production of a constant velocity ratio.

74. Epicycloid and Hypocycloid.—In Fig. 54, let A and B be the centres of motion of the driver and follower respectively, and let $\frac{a'}{a}$ be the required velocity ratio. Divide the line of centres at T, so that $\frac{AT}{BT} = \frac{a'}{a}$. Then, if with radii AT and BT we describe two pitch circles, MN and RS, as shown, these two circles will roll in contact with the required velocity ratio.

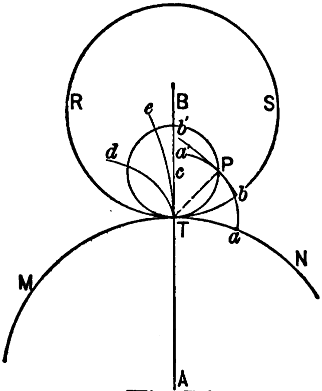


Fig. 54

Let a describing circle be taken of any radius, such as cT, and with it describe an epicycloid Td by rolling it on the *outside* of the pitch circle MN, and a hypocycloid Te by rolling it on the *inside* of the pitch circle RS. If these curves be used for the curved edges of two pieces whose centres of motion are A and B respectively, and the lower one be rotated to the position aa', it will drive the other so as to bring it to the position bb'; for, by the known properties of the curves, they will have their point of contact, P, in the circumference of the describing circle when its centre c is on the line of centres, AB, and they will also have a common normal and a common tangent at that point. Draw the line TP from the point of contact of the two pitch circles to the point of contact of the two curves. Now, on whichever of the two pitch circles we regard the describing circle to be rolling at the instant, its instantaneous centre of motion will evidently be the point T. For that instant, then, the point P revolves about T; that is, it moves in a direction perpendicular to TP, and hence the line TP is the common normal to the two curves at that instant. Of course, this same argument may be applied to any other position of the curves in contact; and, as their normal thus always cuts the line of centres in the fixed point T, it is evident that these curves will transmit motion with a *constant* velocity ratio. Furthermore, as the arc Ta = arc TP, and as the arc Tb = arc TP, we have arc Ta = arc Tb; showing that the velocity ratio will be the same as that of the two pitch circles. By transmitting motion by *sliding contact*, then, between these two curves, we may exactly replace the rolling action of the two pitch circles, as far as the velocity ratio is concerned.

75. Epicycloid and Radial Line.—In Fig. 54 the diameter of the describing circle is less than the radius BT. But this is not a necessary condition. If we change its diameter, we will change the shape of *both* curves; but the two curves generated by the same describing circle will always work together.

If we take the diameter of the describing circle just equal to the radius BT, we will get a special case of the hypocycloid. Under these conditions (Fig. 55) the latter will become a straight line passing through the centre B. All the arguments of the last article apply to this case as well;

and we thus see that in this case an epicycloidal curve turning about *A*, and a radial piece turning about *B*, will, by sliding contact, transmit motion with the same velocity ratio as the pitch circles.

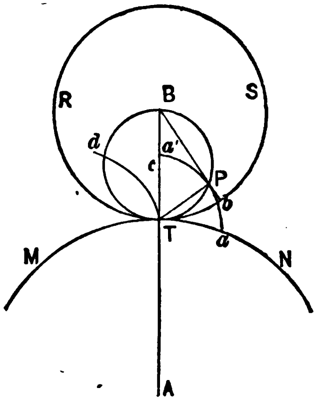


Fig. 55

76. Epicycloid and Pin.—In Fig. 54 the convexity of the two curves lies in the same direction, and they lie on the same side of the common tangent. In Fig. 55 the hypocycloid has become a straight line coinciding with the tangent of the epicycloid. If we increase the diameter of the describing circle still more, the two curves will have their convexities in opposite directions, and they will lie on opposite sides of the common tangent. As the describing circle becomes larger and larger, the hypocycloid becomes more and more convex, and decreases in size, until, when the describing circle is taken with the *same* diameter as the pitch circle *RS*, the hypocycloid will degenerate into a mere point, the tracing point itself. If, then (Fig. 56), we assume a pin to be placed at the point *P* in the circumference of *RS* (the diameter of the pin being so small that the latter may be considered as a mere mathematical line), it follows that it may be driven by the epicycloid *Pa* with the same constant velocity ratio as the pitch circles.

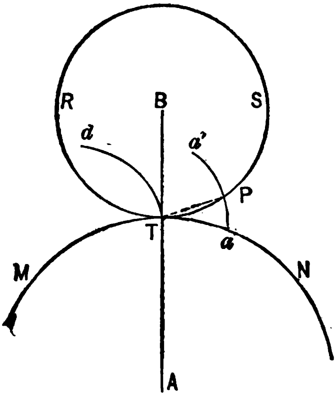


Fig. 56

77. Involutes.—In Fig. 57, let *A* and *B* be the centres of motion, make $\frac{AT}{BT} = \frac{a'}{a}$, and describe the pitch circles *MN* and *RS*, as before. Through *T* draw the straight line *DTE* inclined at any angle to the line of centres; from *A* and *B* drop the perpendiculars *AD* and *BE* upon *DTE*. With these perpendiculars as radii, and *A* and *B* as centres, describe the circles *M'N'* and *R'S'*, which will evidently be tangent to the line *DTE*. Through the point *T* describe the involute *aTd* on the base circle *M'N'*, and the involute *bTe* on the base circle *R'S'*. If these curves be used for the edges of two pieces whose centres of motion are *A* and *B* respectively, and the lower one be rotated to the position *a'Pd'*, it will drive the other to the position *b'Pe'*. For any line tangent to either base circle will evidently be normal to the involute of that circle. Now, when the curves are in contact, the normal to the involute of *M'N'* must be a line drawn from the point of contact tangent to $M'N'$, and the normal to the involute of $R'S'$ must be a line drawn from the point of contact tangent to $R'S'$. But, as the curves must be tangent to each other at the point of contact, they must have a *common* normal at that point. This common normal must evidently be tangent to *both* base circles, and must hence be the line DTE. The point of contact, then, always lies in the straight line DTE; and as the latter is the common normal, and cuts the line of centres in the fixed point T, the velocity ratio is constant, and is equal to that of the *base* circles.

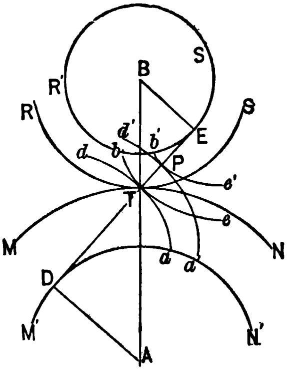


Fig. 57

But, from similar triangles, $\frac{BE}{AD} = \frac{BT}{AT}$; that is, the velocity ratio of the pitch circles is the same as that of the base circles. Hence the involutes, as described, will by *sliding* contact transmit motion with the same velocity ratio as the pitch circles would by *rolling* contact.

78. General Solution.—The four methods just described are the ones most generally employed in the practical solution of the problem of securing a constant velocity ratio in *sliding* contact motions. But we are not by any means limited to the curves above given. Instead of a describing *circle*, we may use a describing *curve* of any shape, provided only that its radius of curvature never exceeds in length the radius of the circle in which the curve is to roll, and thus generate an infinite number of pairs of curves that will satisfy the given condition. Thus, in Fig. 58, let *A*, *B*, and *T* be taken as before, and draw pitch circles *MN* and *RS*. Now, if we take any curve, such as *HTP*, and roll it on the outside of one pitch circle and on the inside of the other, any point of this describing curve will generate two curves which will give the desired velocity ratio by sliding contact. For, let the describing curve be in the position shown, being in contact with the pitch circles at *T*; and let *P* be the describing point. The straight line *TP* will be the common normal to the two curves, because, on whichever of the two pitch circles we regard the describing curve to be rolling at the instant, the point of contact, *T*, is the instantaneous centre of motion; so that the motion of *P* in

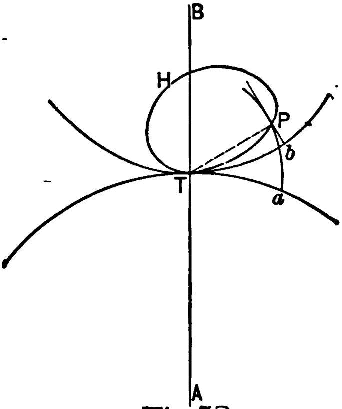


Fig. 58

either curve is perpendicular to *TP*. As the point of contact of the two curves is always in the describing curve, the same argument is true for *any* point of contact. As the common normal will thus always pass through the same point, *T*, of the line of centres, *AB*, these curves will, by moving in contact, produce the desired velocity ratio, exactly replacing the rolling action of the two pitch circles.

79. Conjugate Curves. — Any two curves so related, that, by their sliding contact, motion will be transmitted with a constant velocity ratio, are called conjugate curves. Any curve being assumed at pleasure, we may proceed to find another curve, so that the two curves will be conjugate to each other. If, for instance, in Fig. 58, the curve *Pa* be given, it is only necessary to find the shape of the curve, *HTP*, which, by rolling on the outside of *MN*, will generate *Pa*. By then rolling this describing curve *HTP* on the inside of *RS*, we will obtain the required curve, *Pb*. Again: had *Pb* been given, we could, by a similar process, have found *Pa*; and *Pa* and *Pb* are conjugate curves.

The labor of finding the shape of this describing curve, and using it in this manner, is, however, generally very considerable; so that, for practical purposes, the following simple and satisfactory mechanical expedient, due to Professor Willis, is usually resorted to.

In Fig. 59, *A* and *B* are a pair of boards, whose edges are formed into arcs of the given pitch circles. Attach to *A* a thin piece of metal, *C*, the edge of which is cut to the shape of the proposed curve *ab*, and to *B* a piece of drawing paper, *D*; the curved piece being slightly raised above the surface of the board to allow the paper to pass under it. Roll the boards together, keeping their edges in contact, so that no slipping takes place; and draw upon *D*, in a sufficient number of positions, the outline of the edge *ab* of *C*. A curve, *de*, which touches all the successive lines, will be the corresponding curve required for *B*.

For, by the very mode in which it has been obtained, it will touch *ab* in every position; hence the contact of the two curves *ab* and *de* will exactly replace the rolling action of the two pitch circles. To prevent the boards from slipping, a thin band of metal, such as a watch spring, may be placed between them, being fastened to *B* at *g*, and to *A* at *h*. The respective radii of the circular edges of the boards must, in that case, be made less than those of the given pitch circles by half the thickness of the metal band.

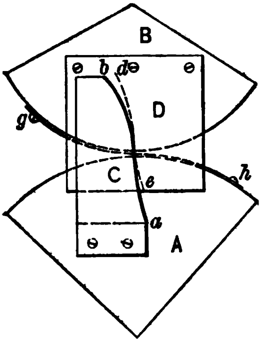


Fig. 59

80. The solutions given above may be used to find the curved edges of any two pieces transmitting motion by sliding contact with a constant velocity ratio, but by far their most important application is in finding the proper shapes for the teeth of wheels.

We shall now give the methods of laying out on paper the principal curves employed for that purpose, and then proceed to examine their practical application in the formation of teeth.

81. Rectification of Circular Arcs.—In constructing these curves, as well as in many other graphic operations, it becomes necessary to determine the lengths of given circular arcs, as well as to lay off circular arcs of given lengths. Either of these problems may, of course, be solved by calculation; but for our purposes it is much more satisfactory to employ the following elegant and surprisingly accurate methods of approximation, devised by Professor Rankine.

I. *To rectify a given circular arc; that is, to lay off its length on a straight line.*

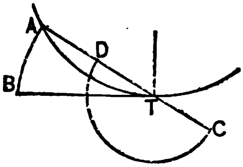


Fig. 60

In Fig. 60, let AT be the given arc. Draw the straight line BT tangent to the arc at one extremity, T. Bisect the chord AT at D, and produce it to C, so that $TC = DT = AD$.

With C as a centre, and radius AC, describe the circular arc AB, cutting BT at B. Then BT is the length of the given arc AT, very nearly.

II. *To lay off, on a given circle, an arc equal in length to a given straight line.*

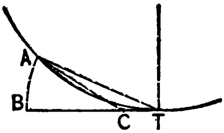


Fig. 61

In Fig. 61, let T be the point desired for one extremity of the arc. Let BT, drawn tangent to the circle at T, be the given straight line. Lay off $CT = \frac{1}{4}BT$. With C as a centre, and radius BC, describe the circular arc BA, cutting the given circle at A. Then the arc AT is equal in length to the given straight line BT, very nearly.

It follows that, to lay off on a given circle an arc equal to a given arc on another circle, we must first rectify the given arc according to I., and then lay off according to II. the required arc equal to the length so found.

82. Degree of Accuracy in Above Processes.—The error in each of these processes consists in the straight line being a little less than the arc. But this difference is very slight, amounting to only $\frac{1}{900}$ of the arc when the latter is 60°. The error varies as the fourth power of the angle, so that it may be reduced to any desired limit by subdivision. Thus, for an arc of 30°, the error will be $\frac{1}{900} \times (\frac{30}{60})^4 = \frac{1}{14400}$. So long, then, as we use these processes for arcs not exceeding 60°, the results will be abundantly accurate for all practical purposes. When the arcs exceed 60°, subdivision should be resorted to.

83. Construction of the Epicycloid.—In Fig. 62, let C be the centre and CT the radius of a circle rolling on the outside of the *fixed* circle whose centre is A and whose radius is AT. Any point in the circumference of the rolling circle will describe a curve, which is known as an *epicycloid.* Let it be required to draw the curve described by the point T of the rolling or describing circle.

Divide the semi-circumference of the latter into any number of equal arcs, $T1'$, $1'2'$, $2'3'$, etc., and through the points of division, $1'$, $2'$, etc., and also through C, describe arcs of circles about A as a centre. Lay off on the fixed circle (Art. 81) the arcs $T1 = T1'$; $1, 2 = 1', 2'$; $2, 3 = 2', 3'$, etc.; and through the points of division, 1, 2, 3, etc., draw radii from A, and produce them.

As the describing circle rolls along the fixed circle, its centre will successively occupy the positions c_1, c_2, c_3, etc. If we draw the describing circle with its centre in any one of these successive positions, as c_2, its intersection b with the circular arc through 2′ will be a point of the epicycloid required. Similarly, we obtain the points *a*, *d*, *e*, *f*, *g*; and the curve drawn through these points will be the epicycloid required. If greater accuracy is required, we need only increase the number of arcs into which we have divided the describing circle.

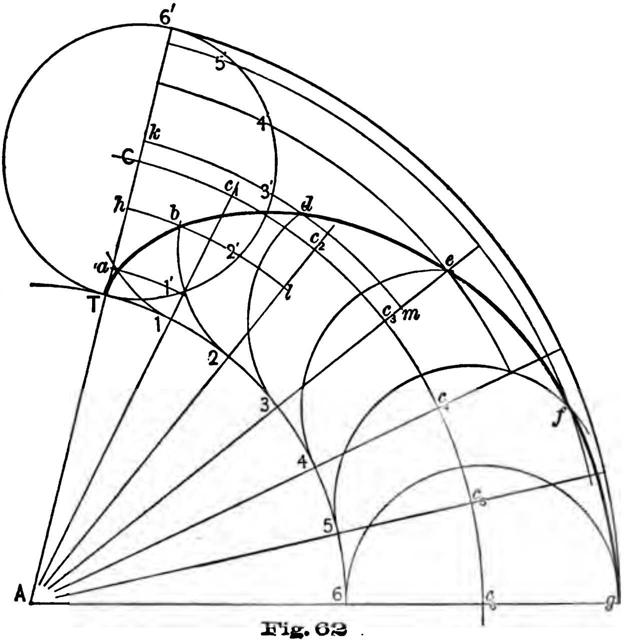


Fig. 62

This method of finding points of the curve is objectionable on account of the resultant obliquity of the intersections at *a* and *f*. This may be avoided, and the construction simplified, by laying off the arc $lb = h2'$, $md = k3'$, etc. In this case it is not necessary to construct the rolling circle in its various positions; and, as this method gives the best results for points of the curve near *T* (which is the part of the curve employed in teeth of wheels), it is greatly to be preferred for practical work.

84. Construction of the Hypocycloid. — The *hypocycloid* is the curve described by a point in the circumference of a circle rolling on the *inside* of a fixed circle. Its construction, shown in Fig. 63, is in every way similar to that of the epicycloid.

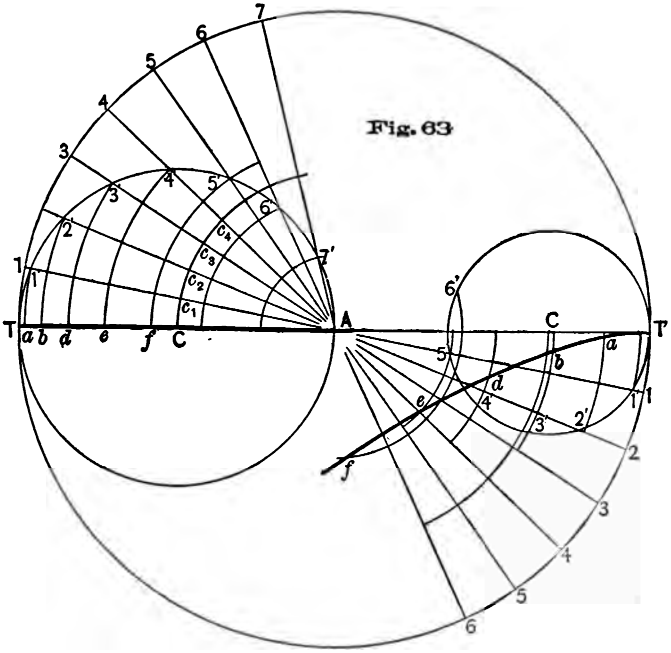


Fig. 63

When the diameter of the rolling circle is less than the radius of the fixed circle, the curve lies on the same side of the centre *A* as the successive points of contact of the two circles. When the diameter of the rolling circle is greater than the radius of the fixed circle, the curve lies on the opposite side of the centre *A*. When the diameter of the rolling circle is *equal* to the radius of the fixed circle, as shown on the left in Fig. 63, the radii $A\,2$, $A\,3$, etc., pass through the points $2'$, $3'$, etc., and the points b and h, k and d, etc., coincide so that the curve becomes a straight line; and this line is a radius of the fixed circle.

85. Construction of the Cycloid.—The *cycloid* is the special case of the epicycloid and hypocycloid, in which the radius of the fixed circle becomes infinite, and the circumference of the circle a straight line. The cycloid is thus described by a point in the circumference of a circle rolling on a straight line. Its construction is in all respects similar to that of the epicycloid and hypocycloid.

86. Construction of the Involute.—The *involute* is generated by a point in a straight line which rolls along a

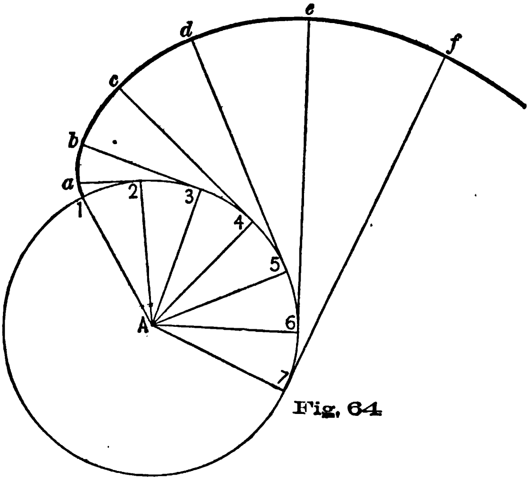


Fig. 64

fixed circle; or we may regard it as formed by the end of a thread which is unwound from about the circle, and kept taut. It will thus always lie in the direction of a tangent to the circle. Hence, to construct the curve, draw any number of tangents to the base circle, and on them lay off the rectified arc of the circle from the point of tangency to the point on the circle where the involute begins. In Fig. 64, then, make $a2 =$ arc 1, 2; $b3 =$ arc 1, 3, etc. The curve drawn through the points 1, a, b, c, etc., will be the required involute.

87. Circular Pitch. — Having divided the line of centres, in any given case, according to the assigned velocity ratio, and described the pitch circles, we must next divide the circumference of each pitch circle into as many equal parts as its wheel is to have teeth. The length of the circular arc measuring one of these divisions is called the *circular pitch*, and often simply the *pitch*, of the teeth. *Circular pitch*, then, is the distance, measured on the circumference of the pitch circle, occupied by a *tooth* and a *space*. This pitch must evidently be the same on both pitch circles. The numbers of the subdivisions, and hence the numbers of teeth, are proportional to the diameters of the pitch circles; and, a fractional tooth being impossible, the pitch must be an aliquot part of the circumference of the pitch circle.

Let $P =$ circular pitch of the teeth in inches;
$D =$ pitch diameter, i.e., diameter of pitch circle in inches;
$N =$ number of teeth;
$\pi =$ ratio of circumference of a circle to its diameter $= 3.1416$.

Then

$$NP = \pi D,$$

and hence

$$N = \frac{\pi}{P} D, \quad D = \frac{P}{\pi} N, \quad P = \frac{D}{N} \pi.$$

From the above relations, we may evidently find any one of the three elements P, D, N; the other two having been given by the problem.

For convenience in calculation, the following table is appended, in which the pitch diameters are calculated for a pitch of one inch.

PITCH DIAMETERS.

FOR ONE INCH CIRCULAR PITCH.

No. of Teeth.	Pitch Diameter.	No. of Teeth.	Pitch Diameter.	No. of Teeth.	Pitch Diameter.	No. of Teeth.	Pitch Diameter.
9	2.86	32	10.19	55	17.51	78	24.83
10	3.18	33	10.50	56	17.83	79	25.15
11	3.50	34	10.82	57	18.14	80	25.46
12	3.82	35	11.14	58	18.46	81	25.78
13	4.14	36	11.46	59	18.78	82	26.10
14	4.46	37	11.78	60	19.10	83	26.42
15	4.77	38	12.10	61	19.42	84	26.74
16	5.09	39	12.41	62	19.74	85	27.06
17	5.41	40	12.73	63	20.05	86	27.37
18	5.73	41	13.05	64	20.37	87	27.69
19	6.05	42	13.37	65	20.69	88	28.01
20	6.37	43	13.69	66	21.01	89	28.33
21	6.68	44	14.00	67	21.33	90	28.65
22	7.00	45	14.32	68	21.65	91	28.97
23	7.32	46	14.64	69	21.96	92	29.28
24	7.64	47	14.96	70	22.28	93	29.60
25	7.96	48	15.28	71	22.60	94	29.92
26	8.28	49	15.60	72	22.92	95	30.24
27	8.59	50	15.92	73	23.24	96	30.56
28	8.91	51	16.23	74	23.55	97	30.88
29	9.23	52	16.55	75	23.87	98	31.19
30	9.55	53	16.87	76	24.19	99	31.51
31	9.87	54	17.19	77	24.51	100	31.83

This table is used in the following manner:—

1. Given the circular pitch and the number of teeth, to find the pitch diameter. Take from the table the diameter

corresponding to the given number of teeth, and multiply this tabular diameter by the given pitch in inches. The product will be the required pitch diameter in inches.

2. Given the pitch diameter and the number of teeth, to find the pitch. Take from the table the diameter corresponding to the given number of teeth, and divide the *given* pitch diameter by this tabular diameter. The quotient will be the required pitch in inches.

3. Given the pitch and the pitch diameter, to find the number of teeth. Divide the pitch diameter by the pitch; and, taking the quotient as a tabular pitch diameter, find from the table the number of teeth corresponding to this tabular diameter. If the latter is not found in the table, the pitch assumed is not an aliquot part of the pitch circumference, and must be altered slightly so as to agree with the number of teeth corresponding to either the next larger or next smaller tabular diameter.

88. Diametral Pitch.—It has been shown in the last article, that the relation between the circular pitch, the pitch diameter, and the number of teeth, introduces the inconvenient number 3.1416. As the number of teeth must be an integer, and as the pitch is usually taken some convenient part of an inch, it follows that the pitch diameter will very often contain an awkward decimal fraction. This may be obviated by the use of the diametral pitch, which is being rapidly introduced in this country.

As the circular pitch is obtained by dividing the pitch *circumference* by the number of teeth, so another ratio may be obtained by dividing the pitch *diameter* by the number of teeth. In practice, it is found more convenient to invert this last ratio; and, when so inverted, it is called the diametral pitch, though theoretically that designation would more properly belong to the ratio as it stood before inversion. In other words, we define *diametral pitch* to be the number of teeth per inch of pitch diameter. Thus, a wheel which has 8 teeth per inch of pitch diameter, is spoken of as an "8-pitch" wheel.

The chief merit of this system, and one which entitles it to great favor, is, that it establishes a convenient and manageable relation between the pitch diameter and the number of teeth; so that the calculations are of the simplest description, and the results convenient and accurate.

Let $M =$ diametral pitch; then we have $MP = 3.1416$, or the product of the circular and the diametral pitches is the number 3.1416.

In this system, the number of teeth and the pitch diameter are so related that the *circular* pitch is usually some decimal; but this is of slight importance, as the circular pitch is rarely set off by actual measurement, but usually by dividing the pitch circle into the required number of parts.

To find the number of teeth in any wheel, multiply the diametral pitch by the pitch diameter. For instance, an 8-pitch wheel of 12 inches pitch diameter has $8 \times 12 = 96$ teeth.

Again: to find the pitch diameter, divide the number of teeth by the pitch. Thus, a 6-pitch wheel of 25 teeth has a pitch diameter of $\frac{25}{6} = 6\frac{1}{4}$ inches.

In the comparison of circular and diametral pitches, the following table will be found useful:—

A	B	A	B	A	B	A	B
¼	12.56	1¾	1.80	3½	0.90	7	0.45
½	6.28	2	1.57	4	0.78	8	0.39
¾	4.20	2¼	1.40	4½	0.70	9	0.35
1	3.14	2½	1.25	5	0.63	10	0.31
1¼	2.50	2¾	1.15	5½	0.58	12	0.26
1½	2.10	3	1.05	6	0.52	16	0.20

Find the given pitch, circular or diametral as the case may be, in column A; then the equivalent pitch in the other system will be found opposite in column B.

In this volume, *circular* pitch is always meant when the word "pitch" is used without further qualification.

CHAPTER VI.

COMMUNICATION OF MOTION BY SLIDING CONTACT.

VELOCITY RATIO CONSTANT.

DIRECTIONAL RELATION CONSTANT.

TEETH OF WHEELS (CONTINUED).

Definitions.—Angle and Arc of Action.—Epicycloidal System.—Interchangeable Wheels.—Annular Wheels.—Customary Dimensions.—Involute System.

89. Teeth. Definitions.—That part of the front or acting surface of a tooth which projects beyond the pitch surface is called the *face*, and that part which lies within the pitch surface is called the *flank*. The corresponding portions of the back of a tooth may be called the *back face* and the *back flank*. The *face* of a tooth in outside gearing is always convex; the *flank* may be convex, plane, or concave. By the *pitch point* of a tooth is meant the point where the pitch line cuts the front of the tooth. In Fig. 72, let the front or acting surface of the teeth be to the left. Then *b*, *k*, are the pitch points of the teeth; *ab* is the *face; bm* is the *flank; de* is the *back face; en* is the *back flank.*

The *depth*, *AD*, of a tooth is the radial distance from root to top; that portion of the top of a tooth which projects beyond the pitch surface is called the *addendum*, *AB*; and a line drawn parallel to the pitch line, and touching the tops of all the teeth of a wheel or rack, is called the *addendum line*, or, in circular wheels, the *addendum circle*, *adA*. The radius of the pitch circle of a circular wheel is called the *geometrical* or *pitch* radius; that of the addendum circle is called the *real* radius; their difference is evidently the addendum.

Clearance is the excess of the total depth above the working depth; or, in other words, the *least* distance between the top of the tooth of one wheel and the bottom of the space between two teeth of another wheel, with which the first wheel gears.

Backlash is the excess of the space between the teeth of one wheel over the thickness of the teeth of another wheel, with which the first wheel gears. The amount of backlash depends on the accuracy with which the teeth are constructed, and should always be made as small as possible. For our present purposes we may neglect it altogether.

90. Angle and Arc of Action.—The angle through which a wheel turns, from the time when one of its teeth comes in contact with the engaging tooth of another wheel until their point of contact has reached the line of centres, is called the *angle of approach;* the angle through which it turns from the instant that the point of contact leaves the line of centres until the teeth quit contact, is called the *angle of recess*. The sum of these two angles is called the *angle of action*. The arcs of the pitch circles which measure these angles are called the *arcs of approach*, *recess*, and *action* respectively. The corresponding arcs must evidently be the same in both pitch circles, while the corresponding angles are proportional to the velocity ratio; in other words, inversely proportional to the diameters of the pitch circles.

In order that one pair of teeth may continue in contact until the next pair begin to act, the arc of action must be at least equal to the pitch arc, and in practice it ought to be considerably greater.

Now, in practice, the friction which takes place between surfaces whose points of contact are approaching the line of centres is found to be of a much more vibratory and injurious character than that which takes place while the points of contact are receding from the line of centres. It is therefore expedient to avoid the first kind of contact as much as possible.

91. Construction of Tooth Outlines. — In Fig. 65, let A and B be the centres of driving and following wheels respectively. Let T be found as usual. Draw the pitch

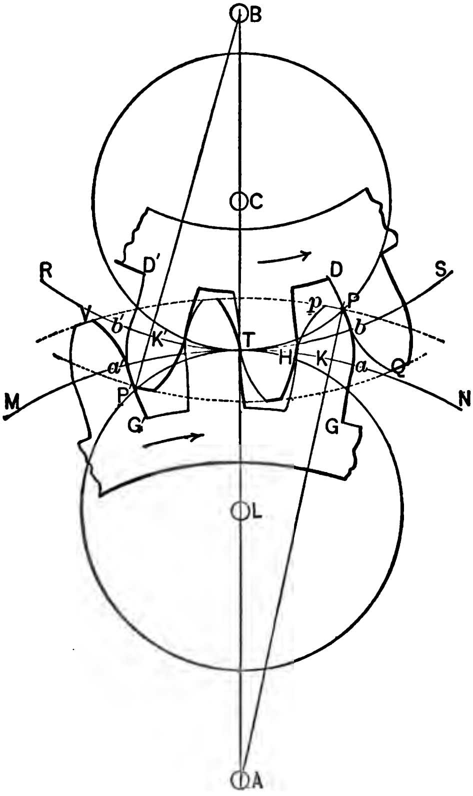


Fig. 65

circle *MN* and *RS*, and assume a describing circle of any radius *CT*, less than $\frac{BT}{2}$. Let *Ta*, *Tb*, be the given pitch arcs, and lay off on the describing circle the arc $TP = Ta = Tb$. If we now roll this describing circle on the outside of *MN*, the point *P* will describe the epicycloid *Pa;* and this curve will be the face of the driver's tooth. Bisect *Ta* at *H*, and draw the epicycloid *Hp*, similar to *Pa*, but reversed in position. Join *pP* by a circular arc concentric with *MN;* then *HpPa* will be the complete outline of that part of the driver's tooth which projects beyond the pitch line.

If we now roll the same describing circle on the inside of *RS*, the point *P* will describe the hypocycloid *Pb*, which will be the acting flank of the follower. But although *Pb* is all of the flank of the follower's tooth that comes in contact with the face *Pa* of the driver's tooth, yet in order to make room for the point of the latter, as it revolves, it is necessary to lengthen the driver's flank. This is usually done by continuing the hypocycloid *bP* to *D*, making the depth of the follower's teeth within the pitch circle *RS* slightly greater than the height of the driver's teeth beyond the pitch circle *MN*.

The bottom of the space between the follower's flanks consists of a circular arc concentric with *RS*.

To determine the faces of the follower's teeth and the flanks of the driver's teeth, we proceed in a precisely similar manner. Assuming a describing circle with radius *TL* less than $\frac{AT}{2}$, and rolling the same on the outside of *RS*, the point *P′* will describe the epicycloid *P′b′* for the face of the follower's teeth. Again, rolling the same circle on the inside of *MN*, the point *P′* describes the acting flank *a′P′* of the driver, which must be extended to *G′*, as in the case of the follower's flanks. By laying off half the pitch arc around the circumferences of both pitch circles, and drawing through these points curves similar to those already found, but alternately reversed in position, and terminating them at the top of the faces and bottom of the flanks by circular arcs concentric with the pitch circles, we will obtain the complete tooth outlines for both wheels.

92. In this construction, the driver's flank first comes into contact with the follower's face at P'. The driver moving as indicated by the arrow, the point of contact travels along the lower describing circle in the arc $P'T$, until it reaches T, where the action between the driver's flank and the follower's face ceases, and that between the driver's face and the follower's flank begins.

The driver still moving as indicated, the point of contact travels along the arc TP of the upper describing circle, and at P the contact ceases.

The points P' and P may be assumed at pleasure on the circumferences of the respective describing circles, and will fix the lengths of the arcs of approach and recess. In the figure, they have been so chosen as to give an arc of approach and an arc of recess each equal to the pitch. These arcs are usually made equal if each wheel is to act indiscriminately as driver or follower; but if the same wheel is always to drive, the arc of recess, for the sake of freedom from vibratory motion, is usually made the greater.

The arc of approach evidently governs the length of face of the follower's tooth, and the arc of recess the length of face of the driver's tooth.

93. Draw the radial line AP, and let K be the intersection of AP with the pitch circle MN. As pointed out by Professor Willis, Ka may be equal to, but can never be greater than, half the thickness of the tooth, as required by the pitch.

In the figure, Ka is less than half the thickness of the tooth. Had the point P been so taken that Ka had been just half this thickness, the tooth of the driver would evidently have been pointed.

94. Size of Describing Circle. — The lengths and shapes of the faces and flanks of the teeth of the wheels, with given arcs of approach and recess, evidently depend on the relation between the diameters of the pitch and describing circles.

If, in Fig. 65, the diameter of the upper describing circle were increased, the face *Pa* would become shorter, and the curvature of both *Pa* and *Pb* would decrease, until, when the diameter of the describing circle became just equal to the radius of *RS*, the hypocycloid *Pb* would become a straight line passing through the centre of the pitch circle *RS* (Art. 75). This fact is often taken advantage of in laying out teeth. When the diameters of both describing circles are thus taken equal to the radii of the pitch circle *in* which they roll, the *flanks* of the teeth of both wheels become *radial* lines, while the faces remain epicycloids. The consequent reduction in the labor of laying out the shape of such teeth has led to their extensive introduction; though, in consequence of the convergence of their radial flanks, they have the disadvantage of being comparatively weak at the root. If the diameter of the describing circle be made still larger, the hypocycloidal flanks will converge still more as they recede from the pitch circle, making the tooth still weaker at the root. Though describing circles have been successfully used having a diameter five-eighths as great as that of the pitch circle in which they roll, yet it seems a good practical rule to make the *radial* flank the limit in this direction. The smaller the describing circles, the longer will be the faces of the teeth, and the greater will be the consequent obliquity of action; but, on the other hand, the stronger will be the tooth. We thus have the two conflicting conditions of obliquity of action and strength of teeth, and the size of the describing circle will be regulated in each case by their relative importance. A good general rule, which is found to work well in practice, is to make each describing circle of a diameter equal to three-eighths of the diameter of the pitch circle in which it rolls.

95. Relation between Pitch and Arcs of Approach and Recess.—The diameters of pitch and describing circles being given, and certain arcs of approach and recess being required, to determine the limits between which the pitch may vary.

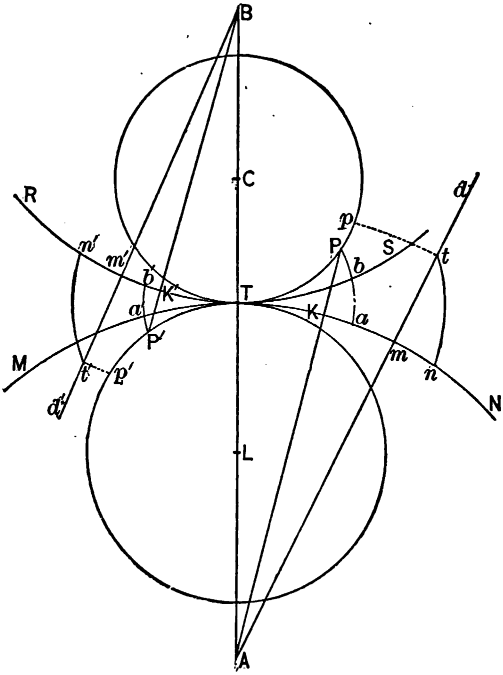


Fig. 66

In Fig. 66, let MN, RS, be the pitch circles, and CT the radius of the upper describing circle. Lay off $Ta = Tb$, the arc of recess desired. Lay off the arc $TP = Tb$, thus fixing the position of P. Describe the epicycloid Pa, and draw

PA, cutting MN in K. Now, as previously explained, if Ka is equal to or less than half the thickness of the tooth, — in other words, if Ka is equal to or less than one-fourth the pitch, — the construction is possible. Hence the pitch of the teeth of the driver must be equal to or greater than four times Ka. If it is just equal to four times Ka, the teeth will be pointed; if greater, they will have some thickness at the top.

Let $Ta' = Tb'$ be the given arc of approach; then, by a similar construction, we find that the pitch of the teeth of the follower must be equal to or greater than four times $K'b'$.

Again: it is evident that the pitch of the driver's teeth cannot be greater than the arc aa'; for, if it were, one pair of teeth would quit contact at P before the next pair would come into contact at P'. Similarly, the pitch of the follower's teeth cannot be greater than the arc bb'. But $aa' = bb' =$ total arc of action. The pitch of the teeth of both wheels must evidently be the same; hence we find, that, to secure the desired arcs of approach and recess, the pitch must *not* be *greater* than the total arc of action, nor *less* than either $4Ka$ or $4K'b'$.

The pitch being given, to find the arcs of approach and recess, draw a radius of MN, and lay off on MN, from the point where the radius intersects the latter, an arc $= \frac{1}{4}$ pitch. Through the point so found draw the epicycloid which would be formed by rolling the describing circle CT on MN, until it meets and intersects the radius at some point. Through this point of intersection draw a circular arc concentric with MN; where the latter cuts the describing circle will be the point P, and the arc of recess will be determined on the supposition that the teeth are pointed. If they are not pointed, let x be the addendum; then a circular arc with radius $AT + x$ will cut the describing circle at the point of quitting contact, P, as before.

The arc of approach is found in a similar manner.

For example, let the pitch and describing circles be given as in Fig. 66, and let the required arcs of approach and recess be $\frac{1}{2}$ inch and $\frac{5}{8}$ inch respectively. Lay off $TP = Ta = Tb = \frac{5}{8}$ inch, and $TP' = Ta' = Tb' = \frac{1}{2}$ inch. Drawing the radial lines AP and BP', we find that $Ka = \frac{11}{64}$ inch, and $K'b' = \frac{3}{32}$ inch. Hence the pitch cannot be greater than $\frac{5}{8} + \frac{1}{2} = 1\frac{1}{8}$ inches, nor less than $4Ka = \frac{11}{16}$ inch. Both of these limiting values of the pitch are, however, to be avoided in practice. For, if the pitch be taken at its smallest possible value, the teeth of the driver will be pointed, and with any wear at the points, the desired arc of recess will no longer be secured; while, on the other hand, if the maximum possible value be given to the pitch, the action will not be smooth, as only one pair of teeth will be in gear at the same time. In addition, the possible values of the pitch will be further limited by the fact that the pitch must be an aliquot part of both pitch circumferences.

Again, let the pitch be given at one inch, and let it be required to determine the maximum arcs of approach and recess. Draw the radius Ad and lay off $mn = \frac{1}{4}$ pitch $= \frac{1}{4}$ inch. Draw the epicycloid nt, and through t describe a circular arc tp concentric with MN; then $Tp = 0.79$ inch is the maximum arc of recess. Proceeding similarly, we find that $Tp' = 0.81$ inch is the maximum arc of approach. But these arcs are determined on the supposition that the teeth of both wheels are pointed. In any practical case, somewhat smaller arcs should be used, so as to give the teeth some thickness at the top.

96. Wheels having Arcs of Recess only. — As previously pointed out, the arc of approach depends on the length of face of the follower's tooth. But from the considerations concerning friction (Art. 90), it is evident that where a very smooth action is required, the arc of approach is objectionable; and in such cases it may be gotten rid of altogether by the simple expedient of cutting off the follower's teeth at the pitch circle.

The follower's teeth, then, having no faces, of course the driver's teeth will need no flanks. In Fig. 67 is shown the construction of a pair of wheels of this kind. The diagram is drawn full size, and is the practical solution of the following problem: —

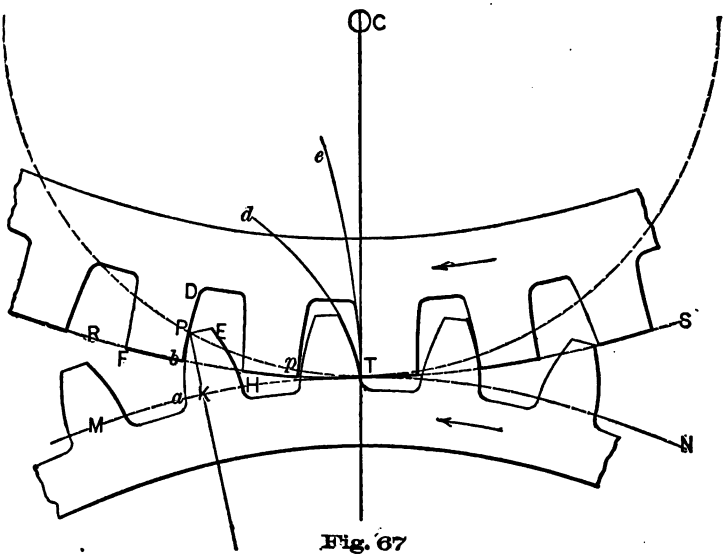


Fig. 67

Distance between centres of pitch circles, 9 inches. Driver (lower wheel) to have 40 teeth; follower, 50 teeth. Arc of recess $= 1\frac{1}{2}$ times the pitch. Divide line of centres AB at T so that $\frac{AT}{BT} = \frac{a'}{a} = \frac{40}{50} = \frac{4}{5}$. Hence the radius of the pitch circle MN = 4 inches, and that of the pitch circle RS is 5 inches.

Let the driver move as indicated by the arrow. Take the diameter of the describing circle $= \frac{3}{8}$ of that of the pitch circle RS of the follower $= \frac{3}{8} \times 10 = 3\frac{3}{4}$ inches. Find the pitch by dividing the circumference of MN into 40 equal parts, and lay off the arc $Ta = 1\frac{1}{2} \times$ the pitch so found. Lay off the arcs $TP = Tb = Ta$, also $Ha = \frac{1}{2}$ pitch. Roll the describing circle on the outside of MN and on the inside of RS, describing the epicycloid Pa and the hypocycloid Pb respectively. Drawing a radial line from P to the centre of MN, we find Ka to be less than $\frac{1}{2}\,Ha$; hence the case is a practicable one.

Through H draw an epicycloid HE similar to Pa, but reversed in position; through P draw an arc of a circle PE concentric with MN, and cutting HE at E. Lay off $bF = Ha$, through F draw a reverse hypocycloid similar to Pb, and join F and b by an arc of the pitch circle RS. Now, Pb is all of the hypocycloid that comes into contact with the epicycloid Pa; but, in order to provide room for the point of the latter, the hypocycloid is continued to D, just as was done in Fig. 65.

If the workmanship were accurate, the wheels would work properly, provided the depth of the space between two successive teeth of one wheel were just equal to the height of the teeth of the other. To provide against any accidental contact, however, both sets of teeth are given *clearance;* that is, the bottoms of the spaces between the teeth are formed by arcs of circles concentric with MN and RS respectively, and at such a distance as to leave a clearance of about one-tenth the pitch in both wheels. The outlines of the teeth are then completed by joining the bottoms of the epicycloids and hypocycloid previously drawn, to these arcs by means of small fillets, as shown in the figure.

The teeth will come into contact at T, the point of contact travelling in the arc TP, until it reaches the point P, where the contact ceases. It is evident that, before any one pair quits contact at P, another pair will have been in contact while the wheels were moving over one-third the arc of action.

97. Wheels with Arcs of Approach and of Recess. — Wheels such as shown in Fig. 67 are sometimes used to great advantage, particularly in light mechanism where smoothness of action is especially important. But whenever the pressure to be transmitted is at all heavy, the wheels should have arcs of both approach and recess, so that more teeth may be in action at the same time. By this means the pressure is distributed over more teeth, while the maximum obliquity of the line of action is not increased. This is, in fact, the form most usually employed in practice, and in Fig. 68 is shown the method of laying out a pair of such wheels. The diagram is drawn full size, and is the practical solution of the following problem: Distance between centres to be 9 inches. The driver (lower wheel) to have 40 teeth, and the follower 50 teeth. Arc of approach to be equal to the pitch, and the arc

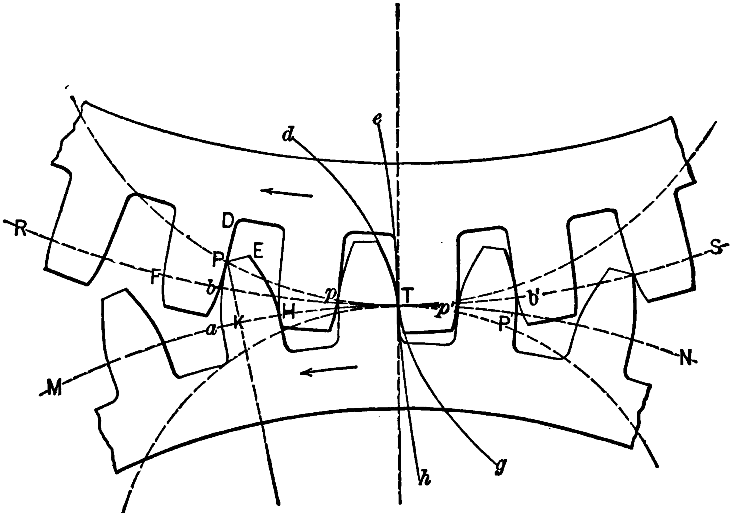


Fig. 68

of recess to be one and a half times the pitch. The conditions given are the same as those given in Art. 96, except that there is to be an arc of approach in this case. The pitch radii of the wheels are 4 and 5 inches, as before; and the diameters of the respective describing circles are 3 and $3\frac{3}{4}$ inches.

The faces of the driver's teeth and the flanks of the follower's teeth are found as in Art. 96, and are, in fact, identical with those there found. In this case, however, we do not finish off the bottoms of the faces of the driver's teeth and the tops of the flanks of the follower's teeth by arcs of circles, as is done in Fig. 67.

Lay off the arc $TP' = Tb'$ = arc of approach. Using the describing circle of three inches diameter, and going through the process explained in Art. 91, we obtain flanks for the driver's teeth, and faces for those of the follower. By this construction, as shown in Fig. 68, there are three pair of teeth in contact; one just quitting contact at P, another in contact at p, and a third pair at p'. In practice, after we have determined that the given arcs of action may be secured with the given pitch (Art. 95), the four curves are usually laid down at T, as shown (Td and Tg being epicycloids, and Te and Th hypocycloids). The *addendum circle* bounding the tops of the teeth, and the *root circle* bounding the bottoms of the spaces, are next drawn. The pitch points of the teeth are then laid off on the respective pitch circles, and the respective curves are drawn through the successive pitch points in alternately reversed directions, being limited at the top by the addendum circle, and connected at the bottom by fillets to the arcs of the root circle.

98. Interchangeable Wheels.—If the describing circle be made of a diameter bearing a fixed ratio to that of the pitch circle, any pair of wheels so laid out will work together; but they cannot both work properly with a third wheel of different diameter. Thus, a given wheel having radial flanks cannot work properly with *two* or more other wheels of different diameters, and also having radial flanks.

If, however, we use the *same* describing circle for *all* the faces and *all* the flanks, we will obtain a series of interchangeable wheels, any one of which will work correctly with any other of the same set. This suggestion is due to Professor Willis, and this method of laying out teeth is invaluable for such purposes as constructing the change-wheels of a lathe.

As, with a *constant* describing circle, the outlines of the teeth will vary with the diameters of the wheels, so as to make the obliquity of action greater as the latter increases, it is usually advisable to employ as large a describing circle as possible. From the considerations discussed in Art. 95, the practical rule follows, that, for a set of interchangeable wheels, the diameter of the constant describing circle should be half the diameter of the pitch circle of the *smallest* wheel of the set.

99. Rack and Wheel. — When a wheel works with a *rack*, the line of centres becomes a perpendicular to the pitch line of the rack, and passing through the centre of the wheel. The rack will travel through a distance equal to the circumference of the pitch circle of the wheel for each revolution of the latter, whatever the number of teeth. The pitch of the rack teeth, therefore, is found by rectifying the pitch arc of the wheel, and laying off this rectified arc upon the pitch line of the rack. In Fig. 69 the two describing circles are made of the same diameter, so that any other wheel of the same pitch whose tooth outlines are formed by means of the same describing circle will also gear with the rack. In fact, the rack is merely a special case of the wheel; and all the deductions of the previous articles as to tooth outlines, arcs of action, etc., apply, with obvious modifications, to this case as well. Both faces and flanks of the rack teeth are *cycloids* (Art. 85): their tops and bottoms are straight lines. The clearance is obtained as usual.

In the figure, which is drawn full size, the diameter of pitch circle of the wheel is four inches, and the wheel has forty teeth. The arcs of approach and recess are each made

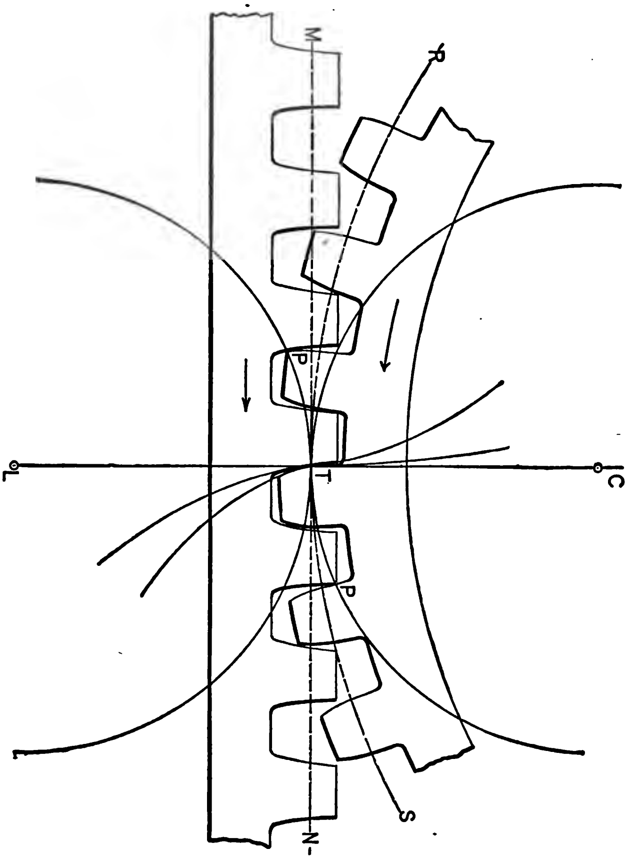


Fig. 69

equal to the pitch. Assuming the *rack* to *drive* to the right, the contact begins at P', the point of contact travelling along the arcs $P'T$ and TP; and at P the action ends.

The principle of making teeth with straight flanks may, of course, be extended to the case of a rack and wheel, as shown in Fig. 70. The describing circle whose diameter is *TB*, the radius of the wheel, generates the *cycloidal faces* of the *rack* teeth and the *radial flanks* of the *wheel* teeth. The radius of the rack being infinite, the diameter of the other describing circle is also infinite; in other words, it is a straight line.

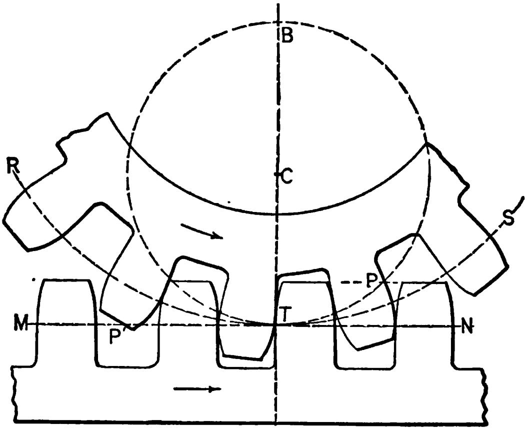


Fig. 70

Hence the *faces* of the *wheel* teeth are evidently *involutes* of the pitch circle, while the *flanks* of the *rack* teeth are straight lines perpendicular to *MN*. The arcs of action and the addendum of the rack teeth are found as before. The *rack driving* to the right, the contact begins at *P′* (the point of intersection of the wheel addendum circle with the line *MN*), travels along the straight line *P′T*, then along the arc *TP* to the point *P* (the intersection of the rack addendum line with the describing circle), where the teeth quit contact. In this form of rack tooth the acting flank has degenerated into a mere point, which is consequently subjected to excessive wear. This is a serious defect, and forms a grave objection to the use of this form of tooth for racks.

100. Annular Wheels. — The construction explained in Art. 91 is applicable not only to the case of wheels in external gear, as there shown, but to that of wheels in internal gear as well. Fig. 71 is drawn full size, and is a practical solution of the following problem: Distance between centres of pitch circles, 3 inches. The pinion to be the driver, and to have 20 teeth; the annular wheel to have 50 teeth. The arc of approach and the arc of recess to be each equal to the pitch. The radii of the pitch circles of the two wheels are evidently 2 and 5 inches respectively. Assuming the diameters of the respective describing circles at $1\frac{1}{2}$ and $3\frac{3}{4}$ inches, we proceed with the construction as before. In fact, on comparing this diagram with Fig. 68, both figures

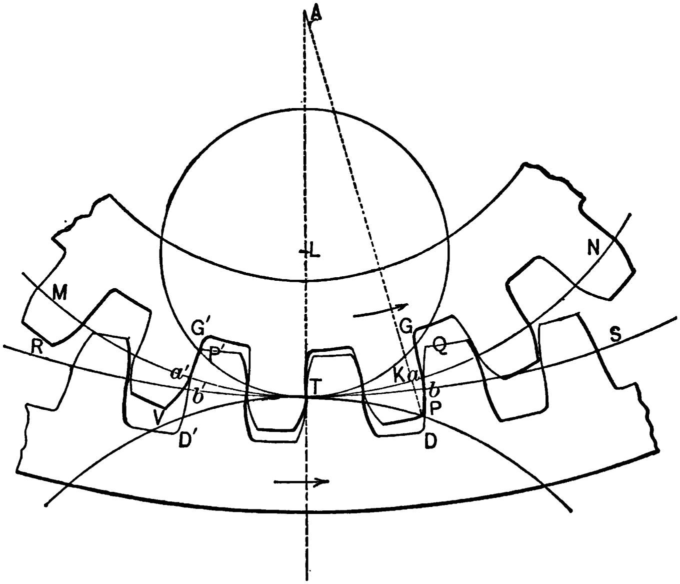


Fig. 71

being similarly lettered, we will see that all the details of construction are the same in both. The pinion is an ordinary spur wheel; while the acting curves of the annular wheel are identical with those of a spur wheel, having the same pitch and describing circles, the tooth of the one corresponding to the space of the other.

The principle of interchangeability (Art. 98) applies to annular wheels just as to spur wheels. Thus, a set of spur and annular wheels may be made in which each spur wheel will gear not only with every other spur wheel, but also with every annular wheel. In this case, however, there must be a difference in the number of teeth of the spur and annular wheels which are to gear together, at least equal to the number of teeth on the smallest pinion of the set.

101. Customary Dimensions of Teeth. — By the preceding methods we may design the teeth of gear wheels so as to fulfil any proposed conditions as to the relative amounts of approaching and receding action. In the majority of cases, however, the precise lengths of the arcs of approach and recess are not a matter of importance; and under these circumstances it is customary to make the whole radial height of the tooth a certain definite fraction of the pitch, the part without the pitch circle being a little less than that within, by which clearance is provided for.

There are a number of such arbitrary proportions; but none of them can be considered absolute, as the proper amount of clearance and backlash evidently depends on the precision with which the tooth curves are laid out, in the first place, and on the accuracy with which the shapes of the teeth are made to conform to the curves so found.

In the manufacture of the best cut gears at the present day, the backs of the teeth barely clear each other when the fronts are in contact; but in the majority of cases a greater allowance is still made, depending for its amount on the accuracy of the workmanship. In cast wheels backlash is absolutely necessary to allow for irregular shrinkage or accidental derangement of the mould.

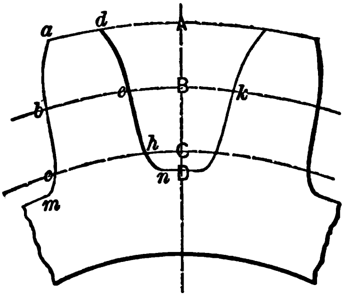


Fig. 72

In Fig. 72, let *bk* = circular pitch = *P*. Then, according to several systems in general use for proportioning teeth, we have the following values: —

Total depth . . AD	$\frac{7}{10}P$	$0.75P$	$\frac{12}{15}P$	$0.750P$
Clearance . . . CD	$\frac{1}{10}P$	$0.05P$	$\frac{1}{15}P$	$0.060P + 0.04$ in.
Working-depth . AC	$\frac{6}{10}P$	$0.70P$	$\frac{11}{15}P$	$0.690P - 0.04$ in.
Addendum, $AB = \frac{AC}{2}$	$\frac{3}{10}P$	$0.35P$	$\frac{5\frac{1}{2}}{15}P$	$0.345P - 0.02$ in.
Thickness of tooth, be	$\frac{5}{11}P$	$0.45P$	$\frac{7}{15}P$	$0.470P - 0.02$ in.
Width of space . ke	$\frac{6}{11}P$	$0.55P$	$\frac{8}{15}P$	$0.530P + 0.02$ in.
Backlash . . $ke - be$	$\frac{1}{11}P$	$0.10P$	$\frac{1}{15}P$	$0.060P + 0.04$ in.

In the first three systems the percentage of backlash is constant, the actual amount of backlash thus increasing directly with the pitch. It seems more rational, however, to make the percentage of backlash greater for small pitches than for large ones; for, the coarser the pitch, the smaller will be the proportion borne to it by any unavoidable error. The last system, that of Fairbairn and Rankine, is founded on this view of the proper proportion of backlash. In this system the percentage of backlash gradually diminishes as the pitch increases. The actual amount as given by this system is, however, rather larger than is generally used at present. Teeth proportioned by any of these systems will in general be of good shape, and answer the purpose desired. Should the wheel have less than about twelve teeth, or should the exact amount of approaching or receding action be of importance, no arbitrary system should be used. In all such cases the proper dimensions of the teeth should be found as previously explained. The backlash and clearance should always be made as small as the character of the workmanship will permit. In our diagrams we have assumed no backlash to exist; but its introduction would have no effect, except to diminish the thickness of the tooth. Instead of half the pitch, as it is in the diagrams, the thickness of the tooth would be half the pitch minus half the backlash. In using the diametral pitch, the working depth of a tooth is almost always taken at two pitch parts of an inch, and the addendum at one pitch part of an inch. That is, in a 4-pitch wheel, the working depth is $\frac{2}{4} = \frac{1}{2}$ inch, and the addendum is $\frac{1}{4}$ inch. The clearance and backlash are taken at from a fourth to an eighth of one pitch part of an inch; thus, in a 4-pitch wheel, they would be taken at from $\frac{1}{4 \times 4}\left(=\frac{1}{16}\right)$ to $\frac{1}{8 \times 4}\left(=\frac{1}{32}\right)$ of an inch. The simplicity of these proportions have led to their almost universal adoption whenever the diametral pitch is employed.

102. Involute System.—It has been shown (Art. 77) that involutes of certain circles possess the property of transmitting motion by *sliding* contact with a constant velocity ratio, and the application of such curves to the formation of the teeth of wheels is shown in Fig. 73.

Let AB be the line of centres, divided at T, so that $\frac{AT}{BT} = \frac{a'}{a}$. Draw the pitch circles MN and RS, and their common tangent $t'Tt$. Draw $p'Tp$, making an oblique angle *pTt* with the tangent *t′Tt*. From the centres *A* and *B* drop the perpendiculars *Ap′* and *Bp* on the line *p′Tp;* and with these perpendiculars as radii, describe the circles *M′N′* and *R′S′*, which will be tangent to the line *p′Tp*.

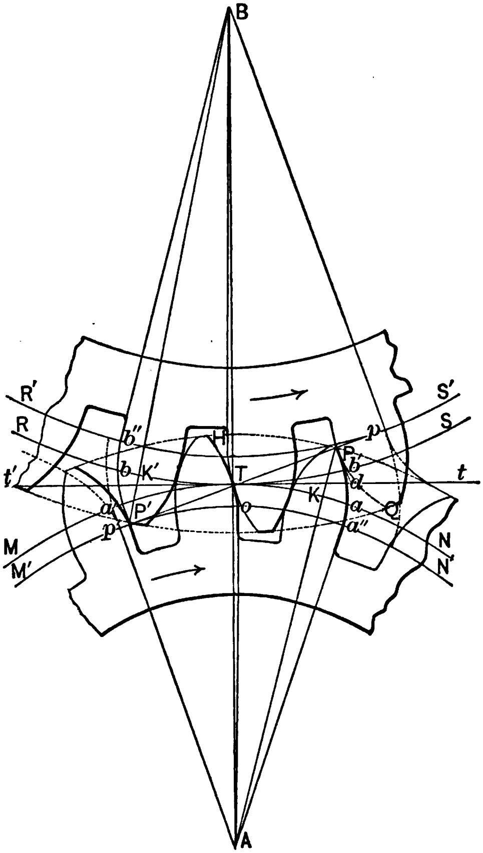


Fig. 73

For the sake of simplicity, let the arcs of approach and of recess each equal the pitch. On the pitch circle *MN*, lay off the pitch *Ta*. With *M′N′* as a base circle, draw the involute *a″P*, passing through the point *a*, and intersecting the line *p′Tp* at *P*. Then *a″P* will be the acting outline of the driver's tooth; and, similarly, *b″P′* will be the acting outline of the follower's tooth. The tooth outlines of each wheel are evidently continuous curves, there being no marked divisions into face and flank, as in the epicycloidal system. Completing the tooth outlines, as shown in the diagram, we find, that, as in the case of epicycloidal teeth, room must be provided for the points of the teeth as they revolve. As the involutes cannot extend within their own base circles, this clearance space is provided by continuing the flanks by radial lines, and joining the latter by means of circular arcs. The contact begins at *P′*, and during the action the point of contact travels along the line *p′Tp* till it reaches *P*, where contact ceases. By making the teeth of both wheels pointed, we can evidently cause them to begin contact at *p′*, and quit contact at *p*.

If this is done, each involute will be long enough to touch the other at its root, and the arcs of approach and recess will be directly proportional to the radii of the pitch circles of the driver and follower respectively. Where pointed teeth are to be employed, it follows that the action will be smoother when the smaller wheel is the driver. But pointed teeth are objectionable here just as in the epicycloidal system; so that, practically, the arcs of approach and recess are adjusted for each particular case, by making the teeth of the proper length.

103. Given the pitch circles, the obliquity of the line of action, and the desired arcs of approach and recess, to find the limiting values of the pitch which will secure these arcs of action. The receding action evidently continues while the point of contact travels from *T* to *P* in the line *TP*, a distance equal to the arc *Oa″*.

The curves *OTH* and *a″aP* being equal involutes of *M′N′*, and the points *T*, *a*, lying in the circumference of the circle *MN*, concentric with that of *M′N′* which contains the points

O, a'', it follows that the angle TAa = angle OAa'', and $\frac{\text{arc } Oa''}{\text{arc } Ta} = \frac{Ap'}{AT}$. Hence $Oa'' = \frac{Ap'}{AT} \times Ta$.

On the tangent Tt lay off the distance $Td = Ta$; from d draw a perpendicular to TP. Then, from the similar triangles TAp' and TdP, we will have $TP = \frac{Ap'}{AT} \times Ta = Oa''$, as required.

Draw the radius PA, cutting MN in K. Now, the pitch cannot be less than $4Ka$. If it is just equal to $4Ka$, the teeth will be pointed; if greater, they will have some thickness at the top. Similarly, the pitch cannot be less than $4K'b$. Hence we find that the pitch cannot be greater than the total arc of action nor less than either $4Ka$ or $4K'b$.

104. Given the pitch circles, the obliquity of the line of action, and the pitch, to find the arcs of approach and recess. From T lay off on the circle MN an arc $= \frac{1}{4}$ pitch, and through the point so found draw an involute of the circle $M'N'$. Through the point of intersection of this involute with the line AB, draw a circular arc concentric with MN, and cutting the line $p'Tp$ at some point P. Then P will be the point at which the teeth will quit contact, and $Ta = \frac{AT}{Ap'} \times TP$ will be the arc of recess. This is only true if the teeth are pointed; if they are not, let x be the addendum. Then a circular arc struck about A, with radius $AT + x$, will cut $p'Tp$ in the point where the teeth quit contact.

The arc of approach is determined in a similar manner.

105. Practical Example.—Fig. 74 is drawn full size, and is the practical solution of the following problem:—

Distance between centres, 9 inches. Driver (lower wheel) to have 40 teeth; follower, 50 teeth. The constant obliquity of the line of action to be 15°. Draw the pitch circles MN and RS with radii of 4 and 5 inches respectively, and their common tangent $t'Tt$. Draw the line of action DE, making the angle $ETt = 15°$. On DE drop the perpendiculars AD and BE, with which, as radii, describe the base circles $M'N'$ and $R'S'$.

The arcs of approach and recess in this problem are each to be equal to the pitch. Hence lay off the pitch arc Ta; and lay off, on the line of action, the distance $TP = \frac{AD}{AT} \times Ta$. Then P is the point at which the teeth quit contact.

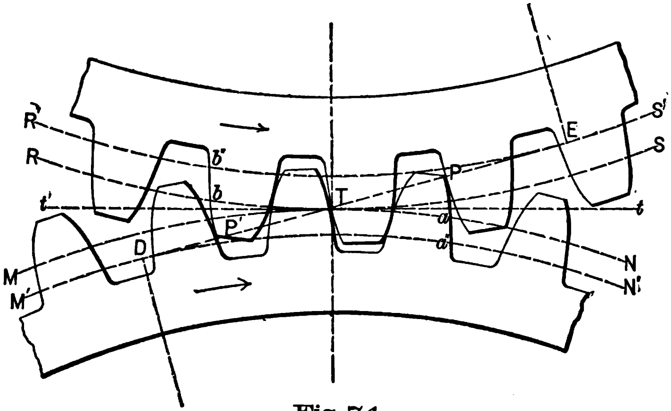


Fig. 74

As the arcs of approach and recess are to be equal, lay off $TP' = TP$. Then P' is the point at which the teeth first come in contact. Through P draw the involute Pa'' of the base circle $M'N'$, and through P' draw the involute $P'b''$ of the base circle $R'S'$. Draw the addendum circles through P' and P, lay off the pitch points of the teeth around the pitch circles MN and RS, and draw through the points so found, in alternately reversed positions, the involutes Pa'' and $P'b''$ respectively.

The tops of the teeth are bounded by arcs of the respective addendum circles. To provide clearance, continue the tooth outlines from the bottoms of the involutes by radial lines to the proper depth. The bottoms of the spaces are circular arcs concentric with the centres of motion, and joined to the tooth outlines by means of small fillets, as shown.

In these wheels, there are evidently always two pairs of teeth in contact. In the position shown, there is one pair in contact at T on the line of centres, while a second pair is quitting contact at P at the same moment that a third pair is engaging in contact at P'.

106. Interference of Involute Teeth. — So long as the teeth are of such a length that the points P and P' (Fig. 74) lie between E and D, they will work properly. In other words, the addendum circle of the teeth of the lower wheel must lie within a circle through E, and concentric with MN. Also, the addendum circle of the teeth of the upper wheel must lie within a circle through D, and concentric with RS. But when the dimensions of teeth are decided on by means of some arbitrary system, such as those of Art. 101, it frequently happens that the length of tooth so found will be great enough to cause the addendum circles to lie *outside* of the concentric circles through E and D respectively. It follows, that the part of the tooth projecting beyond this limiting circle will come into contact with that part of the tooth of the other wheel which lies within the base circle. As this inner part is always made radial, it cannot gear correctly with an involute face, and interference will take place. In case a tooth of such length is considered necessary, and the involute system is to be used, all that part of the face of the tooth of one wheel coming into contact with the radial part of the tooth of the other wheel must be an epicycloid whose describing circle is half the diameter of the pitch circle of the second wheel. As, by this means, we forfeit one of the great advantages of the involute system (the power of varying the distance between centres without affecting the velocity ratio), this construction is not to be recommended, and the length of tooth should not be allowed to exceed the amount determined by the methods of the preceding articles.

107. Rack and Wheel.—If, in Fig. 74, the radius AT of the driver were to *increase*, the curvature of MN, as well as that of the involute of $M'N'$, would necessarily *decrease;* until, when MN became a straight line, $M'N'$ would also become a straight line, and the involute of $M'N'$ would become a straight line, which must be perpendicular to the line of action, DTE.

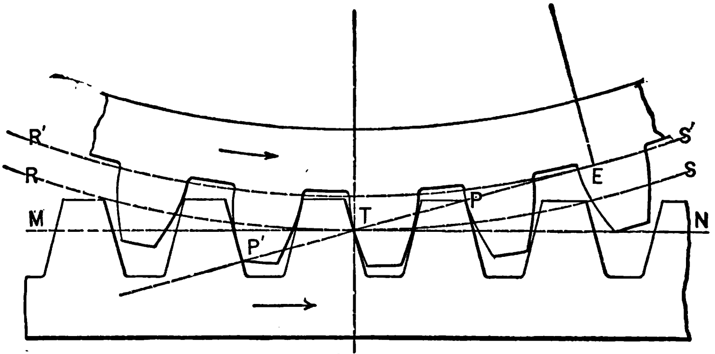


Fig. 75

The method of constructing the teeth is exactly similar to that shown in Fig. 74.

In Fig. 75 the rack is the driver, and the follower is the same wheel that was used as follower in Fig. 74. The teeth of the follower remain the same as in the other case, while those of the rack have straight sides. The tops and bottoms of the rack teeth are straight lines parallel to pitch line MN of the rack. In order to drive the follower through *one* complete revolution, the rack will evidently have to travel a distance equal to the circumference of the pitch circle of the wheel.

The construction of the teeth of *annular* wheels is also in all respects similar to that explained above for spur wheels.

108. Peculiar Properties of Involute Teeth. — In the preceding constructions of practical problems, the line of action was drawn at an angle of 15° with the common tangent of the two pitch circles. This angle is by no means fixed, and may be considerably varied; but experience has shown that for general practice it should *not be greater than fifteen degrees.* As the magnitude of this angle has formed no part of the argument in the preceding cases, it follows that, by varying the obliquity of action, an infinite number of pairs of base circles may be used in connection with any given pair of pitch circles. Conversely, with a given pair of base circles, we may, by altering the length of the line of centres, have an infinite number of pairs of pitch circles. The common tangent to the two base circles will always cut the line of centres into segments having the same ratio as their radii, which will be the same as that of the radii of any of the pairs of pitch circles; from which follow two important practical deductions: —

1. Any two wheels with involute teeth of which the pitch arcs on the *base* circles are equal, will gear correctly with each other.

2. The velocity ratio will not be affected by any change in the distance between their centres.

The peculiarity of interchangeability is also obtainable with epicycloidal teeth under certain conditions (Art. 98). The peculiarity of constant velocity ratio with varying distance between the centres is not found in any other form of teeth, and is of special importance in mechanism requiring exceptional smoothness and uniformity of action. The shafts may be at the proper distance apart, or not, as happens; and they may change position by wearing, or by variable adjustment, as when used on rolls, or they may be brought closer together to abolish backlash. In fact, the involute tooth is remarkably well adapted to such variable demands, and will accommodate itself to errors and defects that are difficult to avoid in practice.

The line of action of epicycloidal teeth is perpendicular to the line of centres at the instant when the point of contact is on that line; but that of involute teeth is constantly in the direction of the common tangent of the two base circles, and hence always oblique to the line of centres. The obliquity of involute teeth, then, is constant; and it is, in general, greater than the mean obliquity of epicycloidal teeth having the same angle of action. The thrust on the bearings is therefore greater with involute than with epicycloidal teeth; and though for heavy pressures this is sometimes a serious objection to the use of involute teeth, yet for ordinary work it would scarcely be so considered.

The involute tooth has a great advantage over the epicycloidal tooth in being of a much stronger shape, spreading considerably at the root, which in the epicycloidal form is often the weakest part. Though the epicycloidal tooth is still in much greater use than the involute tooth, yet the merits of the latter are being rapidly recognized by manufacturers; and, for light work at least, it is gradually coming into more general use to replace the epicycloidal form.

CHAPTER VII.

COMMUNICATION OF MOTION BY SLIDING CONTACT.
VELOCITY RATIO CONSTANT.
DIRECTIONAL RELATION CONSTANT.
TEETH OF WHEELS (CONTINUED).

Approximate Forms of Teeth. — Willis' Method. — Willis' Odontograph. — Grant's Odontograph. — Robinson's Odontograph.

109. Approximate Forms of Teeth. — In order to secure perfect smoothness of action in toothed wheels, it is essential that the tooth outlines should be accurately laid out, as explained in the preceding pages, and that the teeth should be constructed so as to conform exactly with the outlines so found. If the teeth are to be cut, there is no reason why the exact curves should not be used, for it is as easy to form the cutter of the exact shape as of any approximate one; and the cutter once formed, the exact curves can be cut as easily as any other. When the teeth are to be simply cast, however, or when, for other reasons, perfect accuracy is not sought after, we may replace the exact curves by others which approximate to them more or less closely, but which are simpler to construct. When approximate forms of teeth are employed, some one of the arbitrary sets of proportions given in Art. 101 is generally followed.

The two principal methods of approximation are by circular arcs and by curved templets.

110. Willis' Method of Circular Arcs.—In Fig. 76, let A and B be the centres of motion, and T the point of contact of the pitch circles MN and RS. Draw the line of action DTE, making any assumed angle with AB, and erect on it the perpendicular TO. On TO assume the point O, and through this point draw the lines APO and BOQ. We have now formed a system of linkwork, consisting of the arms AP and BQ, connected by the link PQ; and as, by construction, O is the instantaneous centre of PQ, it follows (Art. 25) that $\frac{a'}{a} = \frac{AT}{BT}$ for that instant. If at any point b on DE we draw two curves, abc and hbe, in contact, and of such shape that P and Q are their respective centres of curvature, these curves will, by revolving about centres A and B respectively, produce the constant velocity $\frac{a'}{a} = \frac{AT}{BT}$, the same as that of the pitch circles. In the preceding articles we have already discussed the theoretical shapes of such curves; and, from the above, it is evident that, if circular arcs be drawn through b, with centres P and Q,

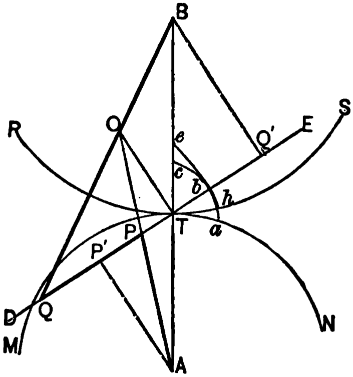


Fig. 76

they will fulfil the required condition *for that instant.* If, however, the teeth are short, and the obliquity is not very great, these arcs differ so slightly from the true curves that they may be substituted for the latter with very good results.

In the figure the arc *abc* will be the face of the tooth of *MN*, while *hbe* will be the flank of the follower.

111. Approximate Involute Teeth by Willis' Method. — In this case the side of the tooth is made to consist of a single arc, and a very simple rule may be obtained.

In Fig. 76, let $TO = \infty$; then AP and BQ will become perpendicular to DE, and the points P and Q will fall at P' and Q' respectively.

Let the circular arcs be struck through T; let R be the radius, AT, of the wheel, and ϕ the angle which DE makes with AB. Then $TP' = R \cos \phi$, which is independent of the wheel RS, as well as of the pitch and number of teeth of MN. If, therefore, the angle ϕ be made constant in a set of wheels, and their teeth be described by this method, any two of them will work together.

Assume $\phi = 75° \; 30'$, which is a very convenient value, for which $TP' = R \cos 75° \; 30' = 0.25038R = \frac{R}{4}$ very nearly.

112. Practical Example. — Let it be required to construct, by this method, the teeth of a wheel of 25 teeth; diameter of pitch circle, 4 inches. Let AT (= 2 inches) be the radius (Fig. 77), and MN the pitch circle, of the proposed wheel. The pitch, as near as may be, is half an inch. We will make the teeth of the proportions given in the first system of Art. 101. This gives addendum = 0.15 inch, total depth = 0.35 inch, backlash = 0.04 inch. Hence draw the addendum and root circles at distances of 0.15 inch without, and 0.20 inch within, the pitch circle, respectively. Draw TP, making an angle of 75° 30′ with the radius, and drop a perpendicular, AP, upon TP $\left(\text{or describe a semicircle upon } AT\text{, and set off } TP = \frac{AT}{4}\right)$; then will P be the centre from which an arc, aTb, described through T, will be the side of the tooth required. To describe the other teeth, draw, with centre A and radius AP, a circle, mn, within the pitch circle MN; this will be the locus of the centres for the teeth. Set off around the pitch circle, arcs of 0.23 inch and 0.27 inch in length alternately, being the respective widths of tooth and space on the pitch circle. Take the constant radius in the compasses, and, keeping one point in the circle mn, step from tooth to tooth, and describe the arcs, as shown in the figure, joining them directly to the arcs of the addendum circle, and by small fillets to the arcs of the root circle. If aTb were an arc of an involute having mn for a base circle, TP would be its radius of curvature at T. These teeth, therefore, approximate to involute teeth; and they

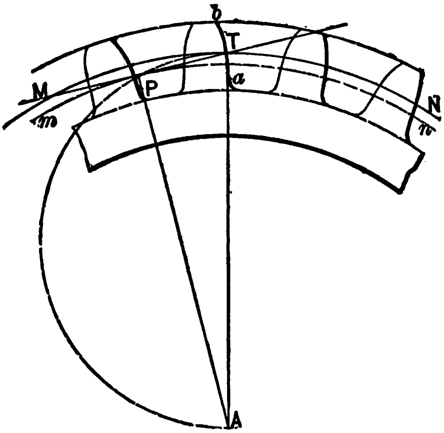


Fig. 77

possess, in common with them, the oblique action, the power of acting with wheels of any number of teeth, and the adjustment of backlash. But, as the sides of the teeth consist each of a *single arc*, there is but one position of action in which the angular velocity is strictly constant; namely, when the point of contact is on the line of centres.

The length of the teeth should always be kept within the limits shown in Art. 102, and in such cases the above method of approximation will give fairly good results. The larger the wheel, the more closely will the circular arcs obtained by this rule agree with the true involute curve.

113. Approximate Epicycloidal Teeth by Willis' Method.—By making the side of each tooth consist of *two arcs* joined at the pitch circle, and struck in such wise that the *exact* point of action of the one shall lie a little before the line of centres, say at the distance of half the pitch, and the *exact* point of the other at the same distance beyond that line, an abundant degree of exactitude will be obtained for all practical purposes.

In Fig. 78, let A and B be the centres of motion, and T the point of contact of the pitch circles MN and RS. Draw DE, making an angle of 75° with AB. This angle is, in fact, arbitrary; but 75° has been found by Professor Willis to give the best form to the teeth.

Draw OTO' perpendicular to DE, and set off the lengths TO and TO', equal to each other, and less than either AT or BT. Through O draw the lines BOQ and APO, and through O' draw the lines $BQ'O'$ and $AO'P'$. By this construction, which is merely an extension of that of Art. 109, we obtain four tooth centres. P will be the centre for the faces of MN, Q the centre for the flanks of RS, Q' for the faces of RS, and P' for the flanks of MN. The *flank* of RS and the *face* of MN will be circular arcs, with centres Q and P respectively, and drawn in contact at a distance of half the pitch to the right of the line of centres; the *face* of RS and the *flank* of *MN* will be circular arcs, with centres *Q′* and *P′*, and drawn in contact at a distance equal half the pitch to the left of the line of centres.

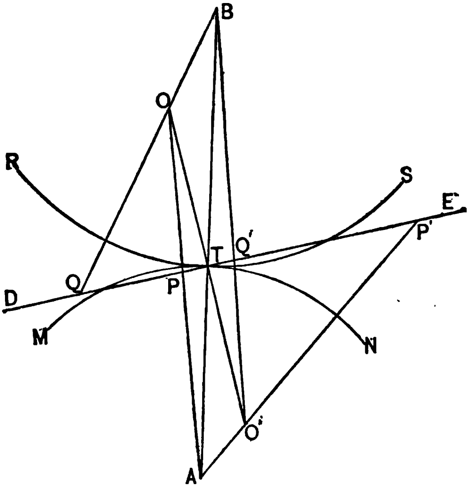


Fig. 78

From the construction it appears that the teeth of one wheel are not changed in shape by any change in the radius of the other wheel. In short, if any number of wheels be described in the above manner, in which the angle *DTA* is constant, the distances *TO* and *TO′* being the same for the whole set of wheels, then any two of these wheels will work together. The distance *TO′* may be determined for a set of wheels by considering that if *A* approach *T*, the point *O′* remaining fixed, *AP′* becomes parallel to *DE*, and the flank of the tooth of *MN* becomes a straight line. If *A* approach still nearer, *P′* appears on the opposite side of *T*, and the flank becomes convex, giving a very awkward form to the tooth. The greatest value, therefore, that can be given to *TO* and *TO′* must be one which, when employed with the smallest radius of the set, will make *AP′* parallel to *DE*. By assuming constant values for this smallest radius, as well as for the angle *DTA*, in a set of wheels, the values of the radii of curvature of the faces and flanks which correspond to different numbers and pitches, may be calculated and tabulated for use, so as to supersede the necessity of making the construction in every case. Thus, the values in the tables of Fig. 79 were obtained by assuming that the least radius was just great enough to give the wheel *twelve* teeth of the required pitch, and that the angle *DTA* was 75°.

114. Willis' Odontograph.—This instrument, represented in Fig. 79, was contrived by Professor Willis for the purpose of laying out the approximate forms of teeth according to the principles of Art. 113. The figure represents the instrument exactly half the size of the original; but, as it may be made of a sheet of bristol-board, this figure will enable any one to make it for use. The side *NTM*, which corresponds to the line *DE* in Fig. 78, is straight; and the line *TC* makes an angle of exactly 75° with it, and corresponds to the radius *AT* of the wheel. This side, *NTM*, is graduated into a scale of twentieths of inches; and each tenth division is numbered, both ways, from *T*.

The instrument is often made of brass, and in that case is of the shape shown in Fig. 80; the tables not being on the instrument, but on a printed sheet accompanying the same.

The manner of using the instrument is shown in Fig. 80. Let it be required to describe the form of a tooth for a wheel of 29 teeth of 3 inches pitch. This determines the radius *AT* of the pitch circle *MN*. Lay off the arcs *TD* and *TE*, each equal to half the pitch, and draw the radial lines *AD*, *AE*. To draw the flank, apply the instrument with its slant edge on *AD*, so that *D* is at the zero point of the scales. In the table headed "Centres for the Flanks of Teeth," look down the column of 3-inch pitch, and opposite to 30 teeth, which is the nearest number to that required, will be found

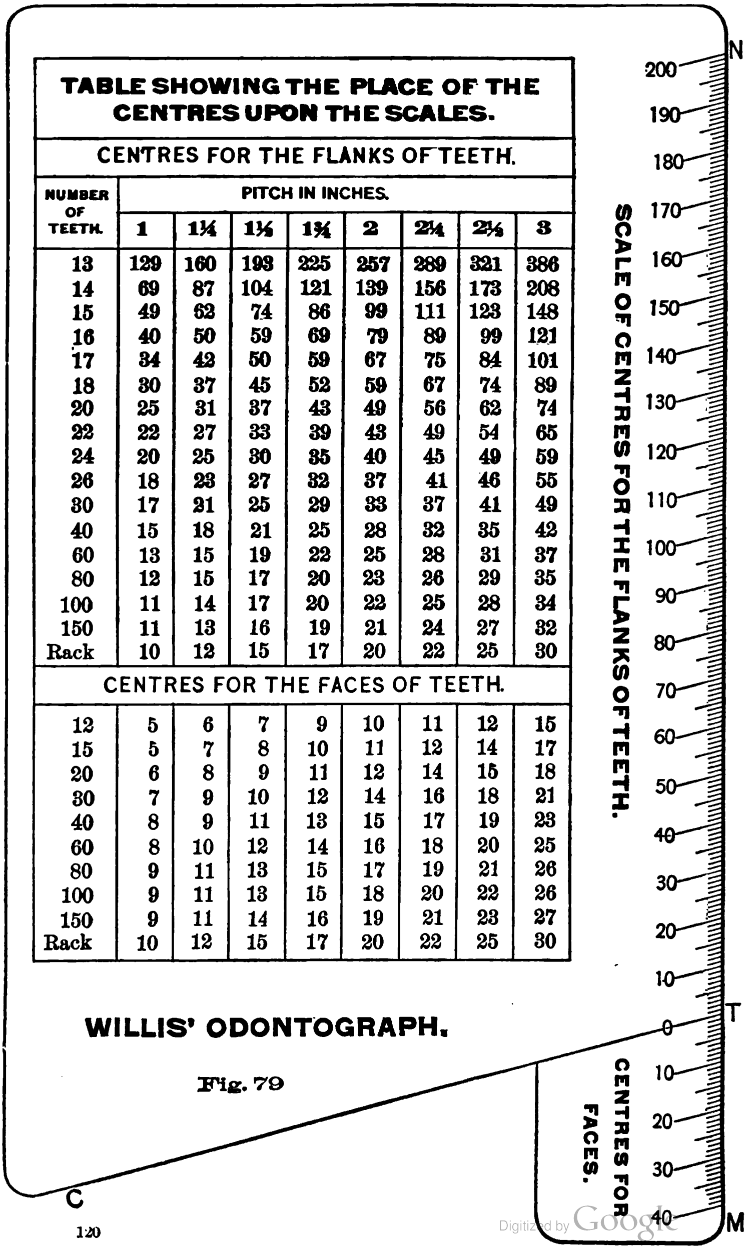


TABLE SHOWING THE PLACE OF THE CENTRES UPON THE SCALES.

CENTRES FOR THE FLANKS OF TEETH.

NUMBER OF TEETH.	PITCH IN INCHES.							
	1	1¼	1½	1¾	2	2¼	2½	3
13	129	160	193	225	257	289	321	386
14	69	87	104	121	139	156	173	208
15	49	62	74	86	99	111	123	148
16	40	50	59	69	79	89	99	121
17	34	42	50	59	67	75	84	101
18	30	37	45	52	59	67	74	89
20	25	31	37	43	49	56	62	74
22	22	27	33	39	43	49	54	65
24	20	25	30	35	40	45	49	59
26	18	23	27	32	37	41	46	55
30	17	21	25	29	33	37	41	49
40	15	18	21	25	28	32	35	42
60	13	15	19	22	25	28	31	37
80	12	15	17	20	23	26	29	35
100	11	14	17	20	22	25	28	34
150	11	13	16	19	21	24	27	32
Rack	10	12	15	17	20	22	25	30

CENTRES FOR THE FACES OF TEETH.

12	5	6	7	9	10	11	12	15
15	5	7	8	10	11	12	14	17
20	6	8	9	11	12	14	15	18
30	7	9	10	12	14	16	18	21
40	8	9	11	13	15	17	19	23
60	8	10	12	14	16	18	20	25
80	9	11	13	15	17	19	21	26
100	9	11	13	15	18	20	22	26
150	9	11	14	16	19	21	23	27
Rack	10	12	15	17	20	22	25	30

WILLIS' ODONTOGRAPH.

Fig. 79

the number 49. The point *g*, indicated on the drawing-board by the position of this number on the scale marked "Scale of Centres for the Flanks of Teeth," is the centre required, from which the arc *Tp* must be drawn with the radius *gT*. The centre for the face *Tn* is found in a manner precisely similar, by applying the slant edge of the instrument to the radial line *AE*. The number 21, obtained from the lower table, will indicate the position, *h*, of the required centre on the lower scale. The arc *Tn* is then drawn, with *h* as a centre, and radius *Th*. We have now the complete tooth outline for one side of one tooth; the curve *pTn* being limited at the top by the addendum circle, and at the bottom by the root circle. Having proceeded thus far, the simplest way of drawing the rest of the tooth curves is to describe two circles about *A*, one through *g* and the other through *h*. Then all the centres for the flanks will lie on the former, and all the centres for the faces on the latter, of these two circles. We may now find these centres by striking from each pitch point an arc with radius equal to *gT* to cut the circle of centres for flanks, and an arc with radius *Th* to cut the circle of centres for faces.

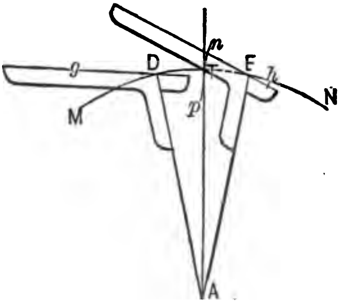


Fig. 80

The curve *nTp* is also correct for an annular wheel of the same radius and number of teeth; *n* becoming the root, and *p* the point, of the tooth. Numbers for pitches not inserted in the table may be obtained by direct proportion from the column of some other pitch; thus, for 4-inch pitch, by doubling those of 2-inch pitch. Also, no tabular numbers are given for 12 teeth in the upper table, because their flanks are radial lines.

The variation in the contour, due to the addition of a single tooth, becomes less and less as the number of teeth increases; so that the same curve will serve for wheels with *nearly* the same number of teeth. Consequently, if the number assigned is not found in the tables, the nearest number found there is to be used instead.

115.* Improved Willis Odontograph.—In Fig. 80 the points g, h, are found by drawing two radial lines, AD and AE, and applying the instrument to each of them, or by drawing two additional lines, gD and Eh, at an angle of 75° with AD and AE respectively, and setting off on them certain lengths obtained from tables. Having found these points, circles of centres are drawn through them, and used as explained above.

If, now, instead of proceeding in this manner, we could find from tables the radii of the two circles of centres, and the radii gT and Th, the construction would be much simplified.

This improvement is due to Mr. George B. Grant, who has calculated the distances of the two circles of centres from the pitch circle, and also the radii of the arcs for the faces and flanks. His results appear in the following table, where "Dis." represents the radial distance between the circle of centres and the pitch circle, and "Rad." the radius of the face or flank arc as the case may be:—

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* The tables in Arts. 115 and 116, and the substance of the matter in those articles, are taken, by permission, from "A Handbook on the Teeth of Gears," by George B. Grant, Boston, Mass.

IMPROVED WILLIS ODONTOGRAPH TABLE.

(Copyright, 1885, by George B. Grant.)

Number of Teeth in the Wheel.		For One Diametral Pitch.				For One-Inch Circular Pitch.			
		For any other Pitch, divide Tabular Value by that Pitch.				For any other Pitch, multiply Tabular Value by that Pitch.			
		Faces.		Flanks.		Faces.		Flanks.	
Exact.	Intervals.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
12	12	2.30	0.15	-	-	0.73	0.05	-	-
13½	13- 14	2.35	0.16	15.42	10.25	0.75	0.05	4.92	3.26
15½	15- 16	2.40	0.17	8.38	3.86	0.77	0.05	2.66	1.24
17½	17- 18	2.45	0.18	6.43	2.35	0.78	0.06	2.05	0.75
20	19- 21	2.50	0.19	5.38	1.62	0.80	0.06	1.72	0.52
23	22- 24	2.55	0.21	4.75	1.23	0.81	0.07	1.52	0.39
27	25- 29	2.61	0.23	4.31	0.98	0.83	0.07	1.36	0.31
33	30- 36	2.68	0.25	3.97	0.79	0.85	0.08	1.26	0.26
42	37- 48	2.75	0.27	3.69	0.66	0.88	0.09	1.18	0.21
58	49- 72	2.83	0.30	3.49	0.57	0.90	0.10	1.10	0.18
97	73-144	2.93	0.33	3.30	0.49	0.93	0.11	1.05	0.15
290	145-rack	3.04	0.37	3.18	0.42	0.97	0.12	1.01	0.13

This improved Willis process will produce exactly the same circular arc as the usual method, with the same theoretical error; but its operation is simpler, and less liable to errors of manipulation. By this process the circles of centres are drawn at once, without preliminary constructions, at the tabular distances from the pitch line; and the table also gives the radii of the face and flank arcs. No special instrument is required, no angles or special lines are drawn to locate the centres, and hence the chance of error is much less.

116.* Grant's Odontograph. — If, in the method described in the preceding article, we use, instead of the circular arcs employed by Professor Willis, arcs which shall approximate still more closely to the true epicycloidal and hypocycloidal curves, we shall evidently obtain more satisfactory results. Mr. Grant has computed and tabulated the location of the centre of the circular arc that passes through the three most important points of the true curve; viz., at the pitch line, at the addendum line, and at a point midway between. The Willis arc runs altogether within the true curve, while the Grant arc crosses the curve twice. The average error of the Grant arc is much less than that of the Willis arc, and it is hence to be preferred.

The circles of centres are drawn at the tabular distances, "Dis.," inside and outside the pitch line respectively; and all the faces and flanks are drawn from centres on these circles, with the dividers set to the tabular radii, "Rad." The tables are arranged in an equidistant series of twelve intervals. For ordinary purposes the tabular value of any interval can be used for any tooth in that interval; but for greater precision it is exact only for the given "exact" number, and intermediate values must be taken for intermediate numbers of teeth.

When the number of teeth is twelve, the flanks are radial, and hence no tabular values are given for the flanks of that number.

To illustrate the use of the following table, let it be required to draw the tooth outline for a wheel of 24 teeth of $1\frac{1}{2}$-inch pitch. Draw the pitch circle with its proper radius of 11.46 inches, and mark off the pitch points of the teeth. Draw the addendum, root, and clearance circles, having fixed on the dimensions of the tooth by means of some system of proportions such as those given in Art. 101.

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* See note on p. 122.

GRANT'S ODONTOGRAPH TABLE.

EPICYCLOIDAL TEETH.

(Copyright, 1885, by George B. Grant.)

Number of Teeth in the Wheel.		For One Diametral Pitch.				For One-Inch Circular Pitch.			
		For any other Pitch, divide Tabular Value by that Pitch.				For any other Pitch, multiply Tabular Value by that Pitch.			
Exact.	Intervals.	Faces.		Flanks.		Faces.		Flanks.	
		Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
12	12	2.01	0.06	–	–	0.64	0.02	–	–
13½	13– 14	2.04	0.07	15.10	9.43	0.65	0.02	4.80	3.00
15½	15– 16	2.10	0.09	7.86	3.46	0.67	0.03	2.50	1.10
17½	17– 18	2.14	0.11	6.13	2.20	0.68	0.04	1.95	0.70
20	19– 21	2.20	0.13	5.12	1.57	0.70	0.04	1.63	0.50
23	22– 24	2.26	0.15	4.50	1.13	0.72	0.05	1.43	0.36
27	25– 29	2.33	0.16	4.10	0.96	0.74	0.05	1.30	0.29
33	30– 36	2.40	0.19	3.80	0.72	0.76	0.06	1.20	0.23
42	37– 48	2.48	0.22	3.52	0.63	0.79	0.07	1.12	0.20
58	49– 72	2.60	0.25	3.33	0.54	0.83	0.08	1.06	0.17
97	73–144	2.83	0.28	3.14	0.44	0.90	0.09	1.00	0.14
290	145–rack	2.92	0.31	3.00	0.38	0.93	0.10	0.95	0.12

From the above table take the values given for the interval 22–24; and, as the pitch is 1½ inches, multiply these tabular values by 1½. We then obtain

Distance between pitch circle and circle of face centres = 0.07;
face radius = 1.08.
Distance between pitch circle and circle of flank centres = 0.54;
flank radius = 2.15.

Draw the circle of face centres 0.07 inch inside the pitch circle, and the circle of flank centres 0.54 inch outside of the pitch circle. With a pitch point as a centre, strike an arc with radius 1.08 inches to cut the circle of face centres, and an arc with radius 2.15 inches to cut the circle of flank centres. With these two points of intersection as centres, describe the face and flank through the pitch point, draw the same arcs in reversed position through a point on the pitch circle whose distance from the pitch point is the desired tooth thickness, connect the faces by an arc of the addendum circle, and join the flanks by fillets to the clearance circle, and the tooth is complete.

This odontograph, as well as Willis', is arranged for an interchangeable set (Art. 98), from a wheel with twelve teeth to a rack.

117. Robinson's Templet Odontograph.—In the use of this instrument, a method entirely different from those just mentioned is pursued. Instead of using circular arcs, the outlines of the teeth are drawn by means of a templet, which is the curved edge of the instrument itself, when the latter is brought into a proper position.

As the epicycloidal curve is normal to the pitch line, and very nearly so to the tangent to the pitch circle drawn from the middle of a tooth, it is clear that if a curve of rapidly changing curvature be so placed as to be normal to the tangent, as above described, and at the same time intersecting the addendum circle at the same point that the epicycloidal curve required for the tooth does, it will represent the epicycloidal tooth face with great precision.

The curve adopted as conforming most closely, in general, with limited initial portions of the epicycloid, is the *logarithmic spiral.* This curve appears to possess the highest degree of adaptation, because of its uniform rate of curvature, and also because this rate can be assumed at pleasure. In adopting the particular logarithmic spiral for the odontograph curve, inasmuch as this spiral may have an infinite variety of obliquities, it is evident that the selection is not a matter of indifference. When the obliquity, or angle between the normal and radius vector, is very small, the arc of this spiral changes curvature less rapidly than when the obliquity is great. When the obliquity is zero the spiral becomes a circle, and when it is 90° the spiral is simply a radius; neither of which approximates to the desired curve.

To find that obliquity which makes the spiral best fit the epicycloid, it will probably be most satisfactory to assume an epicycloid which represents an average of those likely to be used for both curves, and adapt the spiral to it, though any ordinary logarithmic spiral will evidently conform more closely to it than the circle. The spiral which most closely osculates the epicycloid for a pair of *equal* pitch circles is therefore adopted, because the opposite wheel may be either larger or smaller, thus making a higher or lower epicycloid.

By an elaborate mathematical investigation,* Professor Robinson has shown that this curve will produce the required results in all the various cases of epicycloidal and involute gearing.

118. Manner of using Odontograph.—The instrument is shown in Fig. 81 of full size, and of suitable capacity for laying out all teeth below six inches pitch. The curved edge AB is the logarithmic spiral above spoken of; and the curve AC is its evolute, in other words, an equal spiral.

The instrument should be made of metal, because it is intended that it may be used directly for a scribe templet, in which use it will be subject to wear from the passes of the scribe. It has several holes in it, so that it may be attached by wood screws, or by bolts expressly prepared, to any convenient wooden rod, in such a manner, that, when the rod swings around a centre-pin of the wheel, all the faces of the teeth may be described directly from the instrument itself.

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* For the complete mathematical discussion, see No. 24 Van Nostrand's Science Series.

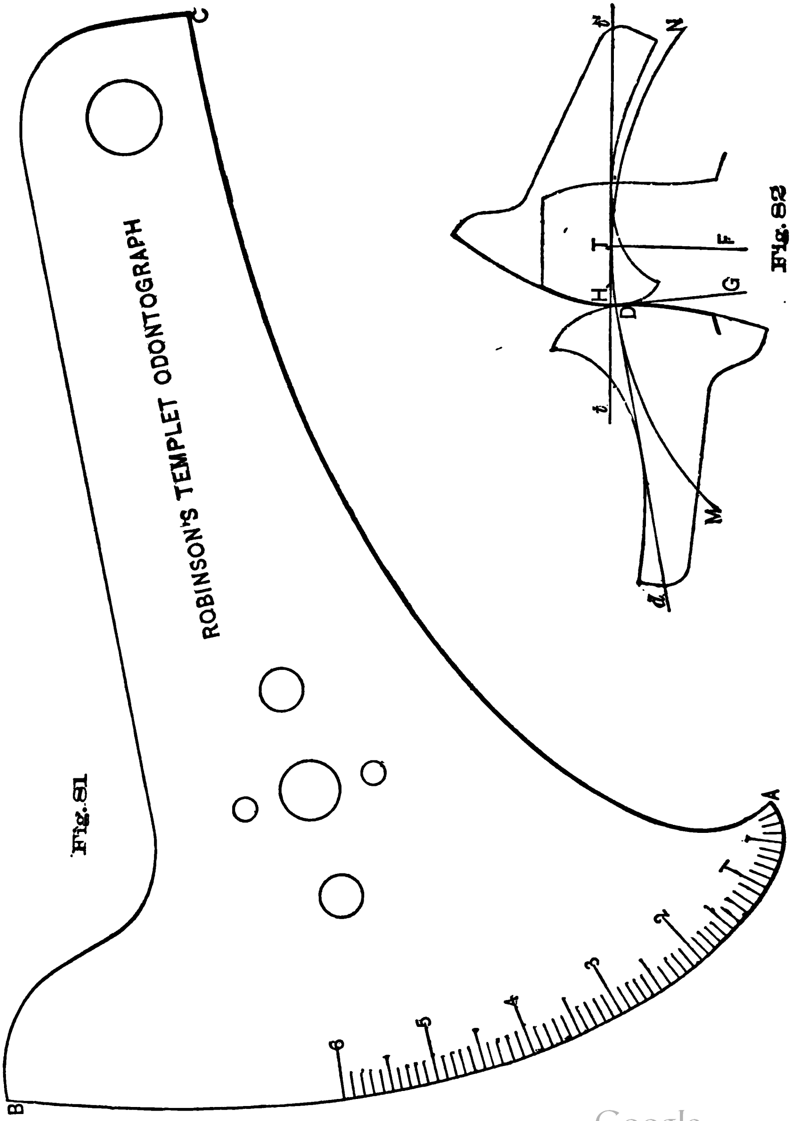


Fig. 81

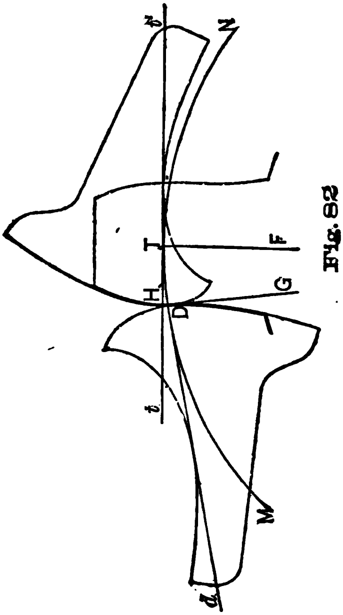


Fig. 82

The desired result is thus obtained directly without the use of a pair of compasses.

Accompanying the instrument are six different tables, varying according to the kind of tooth desired. One of the tables is for the teeth of wheels belonging to an interchangeable series; the other tables are for variously curved flanks and for annular wheels. The manner of using all the tables is nearly the same, so it is simply necessary to indicate the method for any one of them. Fig. 82 shows the manner of using this odontograph to lay out the teeth of a wheel belonging to the interchangeable series.

The table for this system is arranged in four columns, headed respectively, 1, "Diameter in Inches;" 2, "Number of Teeth;" 3, "Face Settings;" 4, "Flank Settings." The two settings are given for one-inch pitch.

In the figure, let *MN* be the pitch circle. If it is not given, it may be found by multiplying the pitch by the number in the column "Diameter in Inches" corresponding to the number of teeth.

Assume the point *T* as the middle of a tooth, and lay off *TD* = its half-thickness. At *T* draw the tangent *tTt'*, and at *D* the tangent *Dd*. Make *TH* = *TD*. Take from the column "Face Settings" the figure corresponding to the number of teeth, and multiply it by the pitch; this will give the setting number. Then place the graduated edge of the odontograph at *H*, and in such position that the number and division of the scale shall come precisely on the tangent line at *H*, while at the same time the other curved edge is tangent to the line *tTt'*. The tooth outline is then traced along the instrument from *D* as far as needed. By turning over the instrument, which is graduated on both sides, and repeating the operation, we get the opposite face of the same tooth.

To draw the flank, find a similar setting number by using the column "Flank Settings." The instrument is to be set with the division at *D*, and the other curved edge tangent to

Dd; and the flank may then be drawn to the proper depth. When it is desired to repeat the operation of drawing the curves all around the wheel, the simplest way to locate the instrument is by drawing circles through the points *A* and *C* when it is once properly located. The instrument can then be readily placed at any tooth outline by placing the graduated edge on the pitch point, and keeping the points *A* and *C* in the circles just mentioned.

For instance, let it be required to draw the teeth of a wheel having 50 teeth of 3-inch pitch. For this number of teeth we find the tabular values: —

Diam. in Inches.	No. of Teeth.	Face Setting.	Flank Setting.
15.917	50	0.42	0.66

The diameter of the pitch circle is $3 \times 15.917 = 47.751 = 47\frac{3}{4}$ inches. The proper setting to draw the face is $3 \times 0.42 = 1.26$, and the corresponding setting for the flank is $3 \times 0.66 = 1.98$.

Hence, to draw the face, the odontograph is placed so that the number 1.26 on the scale is at the point *H* (Fig. 82); and, to draw the flanks, it is placed so as to bring the number 1.98 at *D*.

CHAPTER VIII.

COMMUNICATION OF MOTION BY SLIDING CONTACT.
VELOCITY RATIO CONSTANT.
DIRECTIONAL RELATION CONSTANT.
TEETH OF WHEELS (CONCLUDED).

Pin Gearing.—Low-Numbered Pinions.—Unsymmetrical Teeth.—Twisted Gearing.—Non-Circular Wheels.—Bevel Gearing.—Skew-Bevel Gearing.—Face Gearing.

119. Pin Gearing.—In Art. 76 it has been shown that an epicycloid traced on the pitch circle of the driver, by rolling on the latter a describing circle equal to the pitch circle of the follower, will drive a pin in the circumference of the following pitch circle with the same constant velocity ratio as if the pitch circles rolled together.

In Fig. 83, let MN and RS be the pitch circles. Lay off the equal pitch arcs Ta and Tb; and, with RS as the describing circle, trace through a the epicycloid aD, which will, of course, pass through b. Draw the equal epicycloid TD in reverse position through T, and let D be the point of intersection of the two epicycloids. Then aDT is the complete outline of a tooth of MN which will drive a pin b (having no appreciable diameter) on RS with the constant velocity ratio $\frac{a'}{a} = \frac{AT}{BT}$. Through D draw the arc DP concentric with MN. The point P, where this arc intersects RS, will evidently be the point at which the tooth aDT and the pin *b* will quit contact. Through *P* draw the epicycloid *Pa′* equal to *Da*; then *TP* = *Ta′* = arc of recess. The wheel *MN* moving as indicated by the arrow, the contact will begin at *T*, and the point of contact will travel to the right, along the arc *TP*, until it reaches the point *P*, where contact ceases. The contact is wholly on one side of the line of centres; and when the *teeth drive*, as they should always do (Art. 90), there is *no arc of approach*.

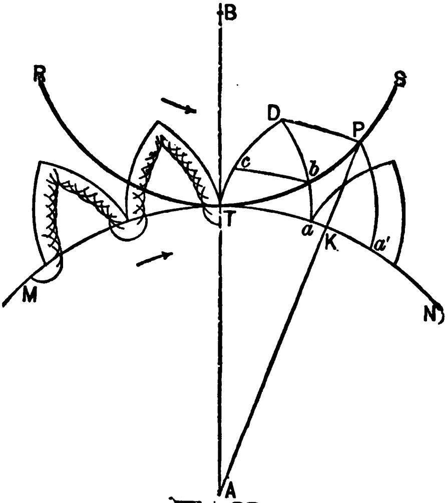


Fig. 83

120. With given pitch circles, to find the relation between the arc of recess and the pitch.

In Fig. 83, let *TP*, the arc of recess, be given. Through *P* describe the epicycloid *Pa′* by rolling *RS* on *MN*; draw the radius *PA*, intersecting *MN* in *K*. Then, in order to secure the desired arc of recess, the pitch must not be greater than *Ta′* = *TP*, nor less than 2*Ka′*. If *Ta* (= 2*Ka′*) be the pitch, and the tooth be pointed, the arc of recess will be

TP, as required. If, with the same pitch, the tooth be given some thickness at the top, the arc of recess will become less; and, when the latter has its smallest value (i.e., when it is just equal to the pitch), the top of the tooth will be cut off so as to give the tooth outline *Tcba.*

121. If the pitch be given, and it is required to find the arc of action which may be secured, lay off on *MN* the given pitch arc *Ta*, and, with *RS* as describing circle, construct the epicycloids *TD* and *aD.* Through their point of intersection, *D*, draw the arc *DP* concentric with *MN.* Then *TP* is the maximum, and *Tb* (= pitch) is the minimum, value of the arc of recess; the tooth in the former case being pointed, and in the latter cut off at *cb.*

122. Pins of Sensible Diameter. — In the preceding articles we have treated the pins as mere mathematical lines; but in practice they must, of course, be given some magnitude, and they are usually made as cylinders of a diameter of about half the pitch. The form of the *tooth* must then be so modified, that, when it acts on the cylindrical surface of the pin, the latter shall move just as though its axis were being driven by the original epicycloid; in other words, the constant normal distance from the latter to the new tooth outline must be equal to the radius of the pin. The manner of finding this derived curve is shown in Fig. 83. About successive points along the epicycloid, as centres, circular arcs are drawn, having the same radius as the pin; a curve drawn tangent to this series of arcs will be the required tooth outline.

In deriving the new tooth outline to act with a pin of sensible diameter, the length of the driver's tooth has evidently been reduced, causing a certain diminution of the arc of recess. Now, assuming the derived curve to be an epicycloid, identical with the original epicycloid, but simply moved in position, it is evident that contact will begin just as the centre of the pin reaches the point *T*; in other words, an arc of approach will have been introduced practically equal to the radius of the pin. Now, although this assumption as to the shape of the derived curve is not strictly true, yet the error thereby introduced is inappreciable, and of no importance in any practical case.

123. Limiting Diameter of Pin.—In the practical construction of problems in pin gearing, it sometimes becomes important to determine the maximum diameter of pin that can be used under given conditions. For instance, let the diameters of the pitch circles and the pitch be given, and let it be required to determine the maximum diameter of pin which will secure a certain arc of action. In Fig. 84, let MN and RS be the given pitch circles, and let the given pitch be $\frac{5}{8}$ inch. Let the required arc of action be $1\frac{1}{2}$ times the pitch; that is, $1\frac{1}{2} \times \frac{5}{8} = \frac{15}{16}$ inch. Lay off the arcs Ta and Tb, each equal to the desired arc of action, and let abe be the epicycloid which would be described by the point b in rolling the circle RS on the outside of MN. Join Tb by a straight line, which will be normal to the epicycloid abe at the point b. Now, as the contact between the derived tooth and the cylindrical pin begins when the centre of the latter is at T, the desired arc of action will be secured if contact ceases when the centre of the pin reaches b. Hence b will be the position of the centre of the pin at the moment of quitting contact, and the point of the tooth must evidently lie on the line Tb at a distance from b equal to the radius of the pin.

But as ac is the pitch arc, the point of the tooth must evidently also lie on a radius bisecting this pitch arc; and it will consequently be found at P, the intersection of these two lines. Pb then will be the radius that will secure the given arc of action on the supposition that the teeth are pointed.

But if, as is usually the case, it is desired that the teeth should have some thickness at the top, the radius of the pin must be made somewhat smaller than the maximum radius just found.

Thus, in the present example, let us determine the shape of the tooth which will give the same arc of action with a smaller pin. Let the pin be made of the usual diameter employed in practice; namely, ¼ pitch.

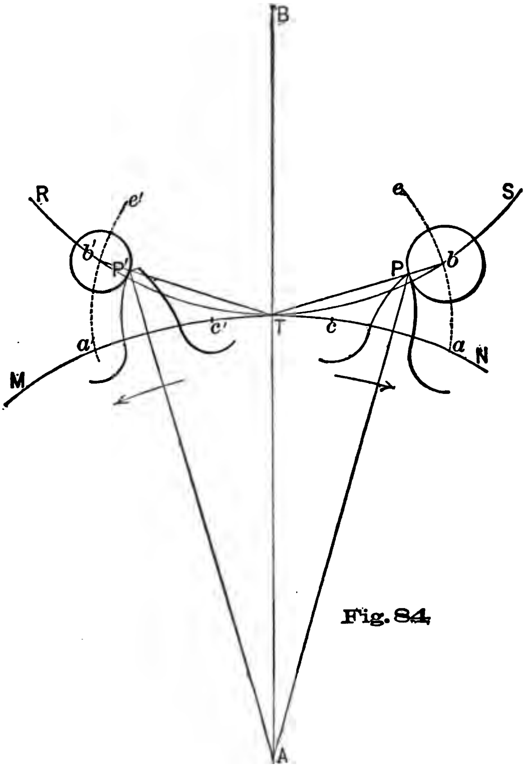


Fig. 84.

In Fig. 84, on the left of the centre line AB, lay off $Ta' = Tb' =$ arc of action, and $a'c' =$ pitch, as before. About b' as a centre, describe the circle of the pin with a radius, $b'P'$, of one-fourth the pitch. Now, although P' will evidently be the extremity of the derived tooth curve, yet it will be found that the radius passing through that point will no longer bisect $a'c'$. In fact, the tooth now has some thickness at the top; and drawing, in a reversed position on the other side of the middle point of $a'c'$, the derived curve found for this diameter of pin, and joining the two curves at the top by a circular arc through P' and concentric with MN, we have the complete tooth outline for this case.

124. Practical Example.—In Fig. 85 is shown a practical example of pin gearing, the diagram being drawn full size, and being the solution of the following problem:—

Distance between centres of pitch circles, 9 inches. Driver to have 50 teeth; follower 40 pins. Arc of action to be $2\frac{1}{4}$ times the pitch. Dividing the line of centres at T in accordance with the given number of teeth, we find the radius of the driver to be 5 inches, and that of the follower to be 4 inches.

Rolling the circle RS as a describing circle on the outside of MN, the point T will describe the epicycloid Tt, which which will be the form of a tooth of MN that would work with a pin of no appreciable diameter on RS. To find the maximum diameter of pin that will secure the desired arc of action, we proceed as in Art. 123. Lay off the arcs Ta and Tb, each equal to the required arc of action, i.e., $2\frac{1}{4}$ times the pitch; and lay off aa' equal to the pitch. Joining Tb, and drawing the radius bisecting aa', we find the maximum size of the pin and the corresponding shape of the tooth, as shown in dotted lines. We may, of course, use any radius of pin less than Pb, and still secure the desired arc of action. In order to get teeth of better proportions, let us make the radius of the pin equal to one-fourth the pitch. On making a construction similar to the one explained in the latter part of Art. 123, we will find the shape of the tooth as finally drawn in the diagram. Just as in other gearing, clearance must be given at the bottom of the spaces, and this is usually done by means of circular arcs, as shown.

The principal advantages of pin gearing are its smoothness of action, and the facility with which the pins may be turned in a lathe.

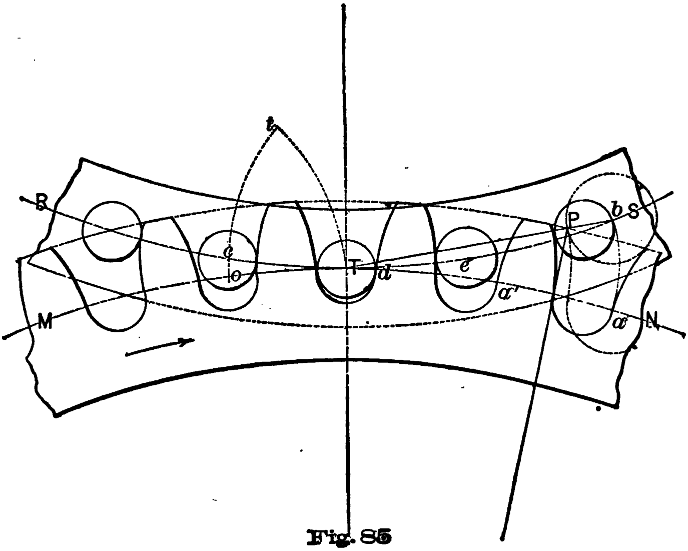


Fig. 85

The pin wheel is often made of two plates, the ends of the pins being fixed into equi-distant holes in both plates, thus making a very strong arrangement, and one which is frequently employed in clock-work. Such wheels are called lanterns or trundles, and their pins are called staves.

125. Rack and Wheel. — As previously stated, the *pins* are always given to the follower, and hence this combination will present two cases according to whether the rack is driver or follower. In Fig. 86 the rack drives and the wheel carries the pins. The teeth of the rack are formed by curves parallel to the *cycloids* which would work correctly with the axes of the pins. In Fig. 87 the wheel drives and the rack carries the pins. The teeth of the wheel are formed by curves parallel to the involutes of its own pitch circle, which would work correctly with the axes of the pins.

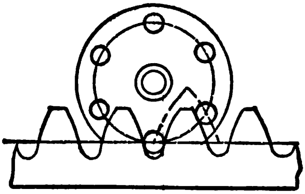


Fig. 86

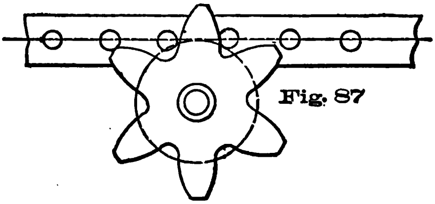


Fig. 87

126. Annular Wheels. — If the annular wheel drives, as in Fig. 88, the pins are given to the small wheel, and the teeth of the annular wheel are formed by curves parallel to the hypocycloids which would work correctly with the axes of the pins. If the annular wheel is the follower, as in Fig. 89, it carries the pins; and the teeth of the small wheel are formed by curves parallel to the epicycloids which would work correctly with the axes of the pins.

When the annular wheel is the driver, and is twice as large as the wheel with which it gears, the hypocycloids become straight lines, and the parallel tooth outlines will evidently also be straight lines.

Fig. 90 shows such an arrangement, in which the pin wheel has but three pins, while the wheel teeth are formed by cutting three straight grooves, intersecting each other at the centre of the wheel, at angles of sixty degrees, each being of a width equal to the diameter of a pin. By placing rollers on the pins, and making the widths of the slots equal to the diameter of these rollers, this arrangement of pin

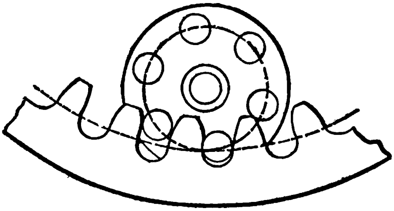


Fig. 88

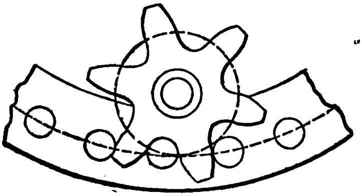
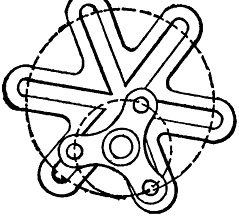


Fig. 89

Fig. 90



gearing can be used as a shaft coupling to drive in either direction.

127. Low-Numbered Pinions. — As the number of teeth in a wheel decreases, the teeth themselves become longer, and both the obliquity of action and the amount of sliding rapidly increase. Pinions having very few teeth are, for these reasons, unsuitable for general use; and accordingly we find that in practice no wheel of less than about twelve teeth is employed if it can possibly be avoided. In order to secure smoothness of action and a minimum obliquity of pressure, the number of teeth assigned to any given wheel is usually so great that no doubt exists as to their successful working. It occasionally happens, however, that it becomes imperatively necessary to employ wheels having as few teeth as possible; and it then becomes a matter of importance to determine whether the desired numbers of teeth will work together.

128. Practical Example. — The practicability of any assumed case can be readily determined by the construction of a diagram, keeping in mind the limitations as to pitch, arc of action, etc., explained in previous articles. For example, let us examine the case of two pinions, of five and seven teeth respectively, and having radial flanks.

In Fig. 91, let A and B be the centres of the equal pitch circles, and T their point of tangency. As the flanks are to be radial, the diameters of the describing circles will be equal to the radii of the respective pitch circles, as shown. Assume the arc of action to have its smallest value (namely, just equal to the pitch arc), and let the arcs of approach and recess be equal. Constructing the teeth under these conditions (Art. 97), we will obtain the wheels shown in Fig. 91. These wheels will just barely work, one pair of teeth quitting contact at P at the same instant that another pair are coming into contact at P'. It is evident that, by continuing the opposite faces until they meet, the arc of action can be some-

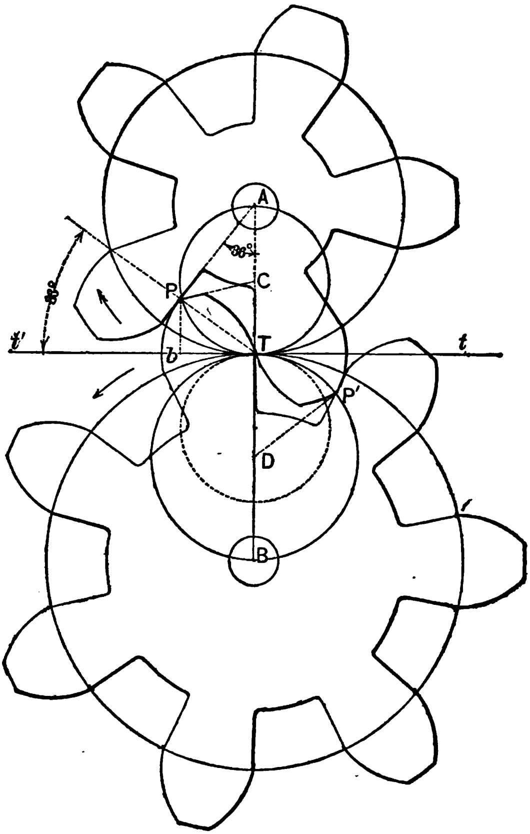


Fig. 91

what increased without making any other change. Two such wheels, then, can be made to work, though they will never run with the smoothness of action that characterizes wheels having a large number of teeth.

129. In Fig. 91, let PT represent in magnitude, as it does in direction, the pressure between a pair of teeth at the moment of quitting contact, the angle PTt' being thirty-six degrees. Then Pb evidently represents the component of PT which tends to force the axes apart, thus producing friction and wear in the bearings. Now, calling p the tangential pressure necessary to transmit any given power, it is evident that in this case $PT = p \times \frac{AT}{AP} = 1.24\ p$, and $Pb = PT \sin 36° = .73\ p$. The obliquity has thus caused a pressure between the axes almost three-fourths as great as the pressure producing rotation. It is evident that there is a limit beyond which the angle of maximum obliquity cannot go without increasing the prejudicial component Pb to an inordinate extent. The limit for the mean obliquity is usually placed at fifteen, and that for the maximum, at thirty degrees; though, where the pressure to be transmitted is not great, thirty-six degrees may be made the limit for the latter. In the case of involute teeth, the obliquity is *constant*, and should never exceed fifteen degrees.

The maximum diameter of the describing circle (Art. 94) is usually taken as half the diameter of the pitch circle *in* which it rolls; but, for special reasons, it may be increased to five-eighths of that diameter. By repeating the construction within these limits of obliquity and of size of describing circle, we will find that *five* is the least number of teeth for each of two equal pinions, that *four* will work with *five* or any greater number, that *three* will work with any number greater than *fourteen*, and that less than three cannot be made to work at all.

130. Two-Leaved Pinion. — There is, however, an exception to the last statement; for if the teeth are placed in *parallel* planes, instead of, as usual, in the *same* plane, a two-leaved pinion can be made to drive in a very satisfactory manner.

This arrangement is shown in Fig. 92. *B* represents a disc, to which teeth *b*, *b′*, *b*, *b′*, etc., are fixed alternately on opposite sides. The acting surfaces of these teeth are straight, and radiate in direction from the centre of *B*. The driver is formed of a pair of double epicycloids, of which *A* is in the plane of the teeth *b*, *b*, etc., and *A′* is in the plane of the teeth *b′*, *b′*, etc. The radius of the pitch circle of *B*, and hence the diameter of the describing circle for the teeth, is equal to the distance from the centre of *B* to the outer extremity of one of its teeth.

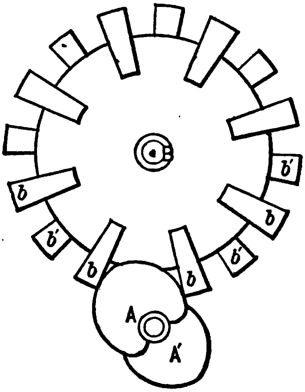


Fig. 92

The action is not very oblique, but the amount of sliding is considerable. As the driver has only faces, and the follower only flanks, the action takes place on one side of the line of centres only. The combination is always used so that the action may be receding; and the result is, that the motion is exceptionally smooth and noiseless.

A pinion of one tooth communicating a constant angular velocity ratio between parallel axes, appears absolutely impossible. The endless screw is equivalent, however, to a pinion of a single tooth.

An arrangement similar to that of Fig. 92 may also be employed to give motion to a rack, as shown in Fig. 93. The construction in this case is simplified, the curves of the pinion becoming involutes of the pitch circle, while the rack consists of a straight bar, having equal rectangular pieces fixed to it at regular intervals on both sides.

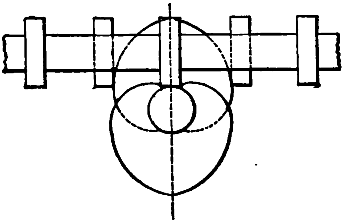


Fig. 93

This arrangement is objectionable on account of the wear being confined to a single point, just as was explained in the case of Fig. 70. Hence, in any practical case it would be better to use a describing circle of comparatively small diameter, making the outline of the rack teeth cycloidal in shape, and thus distributing the action over a greater amount of surface.

131. In Arts. 128 and 129 we have assumed the arcs of approach and of recess to be equal; that is, each equal to half the pitch. Though the total arc of action cannot be less than the pitch, yet we may evidently vary the relative amounts of approaching and receding action at pleasure. If, as is usually the case, the arc of recess is to be the greater, it is evident that a pinion of fewer teeth can be used to drive than to follow; for the arc of recess depends upon the length of the driver's teeth, and this length again depends on the size of the describing circle. If we take a given wheel and pinion, and gradually decrease the number of teeth in the pinion, — in other words, make it of a smaller diameter, — we shall evidently also decrease the diameter of the describing circle, which generates the faces of the wheel teeth, and hence diminish their lengths. The generating circle for the faces of the pinion's teeth will not be affected, so that the length of these teeth will remain almost the same.

But, whatever the conditions, we can evidently determine the practicability of any given case by the construction of a diagram, as explained.

132. Least Follower for given Driver. — Though the construction of a diagram will always enable us to ascertain whether a given combination of driver and follower is possible, yet the problem may sometimes be stated in a more general form. For example, let it be required to find the least number of teeth that can be given to a wheel which is to follow a given driver. Let the pitch circle, pitch, and arc of recess of the driver be given as in Fig. 65. Taking the upper describing circle, as given in the diagram, it is evident that the follower may have less teeth than there shown. For we may make the radius of the follower's pitch circle only twice TC, in which case the follower's teeth will have radial flanks.

Now, if the driver's teeth were to be pointed, instead of as shown in the diagram, it is clear that with the given pitch, the required arc of recess could be secured with a smaller describing circle; and hence a smaller follower could be used. In this case, at the moment of quitting contact, the point of the driver's tooth must lie at the intersection of the upper describing circle with a radius of MN bisecting Ha, and at a distance from T, measured on the circumference of the describing circle, equal to the arc of recess. Hence, drawing the radius bisecting Ha, and finding (Art. 81) the position of this point, we describe through the latter and the point T a circle whose centre lies on AB. This will be the upper describing circle required. If we then make the diameter of the follower's pitch circle twice the diameter of the describing circle thus found, we shall evidently have determined the required least number of teeth, on the assumption that they are to have radial flanks. As previously mentioned, the diameter of the pitch circle may, for special reasons, be made as small as $\frac{5}{8}$ of that of the describing circle; but the size of this pitch circle must always be such that the given pitch will be an aliquot part of the circumference. Having thus determined the least number of teeth, we must ascertain if the obliquity is within the desired limits. If it is found to exceed the assigned limit, the size of the pitch and describing circles must be increased until the obliquity is reduced to the proper amount.

133. Again, the assigned conditions may belong to the follower, and it may be required to determine the least number of teeth for the driver. This case may be solved by an obvious modification of the above process. Both of these cases will also present themselves in the involute system. With involute teeth, the maximum arc of recess will evidently be secured if the driver's tooth be pointed, and if its point touch the base of the follower's involute at the moment of quitting contact. The obliquity of action which secures this result is the greatest possible, and gives the minimum *follower* that can be employed.

Similar problems will occur in regard to annular wheels; and such problems may be solved by similar methods, the only peculiarity being, that there will be both maximum and minimum values.

The *least* annular wheel which can *be driven* by a given pinion must have *one* tooth more than the pinion. The smallest pinion which can be thus used is one of three teeth, the wheel then having four teeth.

The *least* annular wheel that can *drive* a given pinion must have one and a half times as many teeth as the pinion when the latter has radial flanks. The various questions of limiting numbers may also be solved in pin gearing, though the method is more complex, owing to the peculiar nature of the derived curve.

134. But, in any case, it is simply a question of graphical construction; the teeth being laid out in accordance with the prescribed conditions. Tables have been prepared giving such least numbers, calculated with considerable exactness, for various arcs of recess; and though it is always preferable to make the graphic construction for the special case under consideration, yet the following brief extract from such tables may not be without interest. In these tables the flanks of all the spur wheels are supposed to be radial, and the thickness of the tooth and the width of the space, measured along the pitch circle, are supposed to be equal.

EPICYCLOIDAL GEARING.

TABLE OF THE LEAST-NUMBERED SPUR WHEELS AND GREATEST-NUMBERED ANNULAR WHEELS THAT WILL WORK WITH GIVEN PINIONS.

	Number of Teeth in Given Pinion.	Least Number of Teeth in Spur Wheel		Greatest Number of Teeth in Annular Wheel	
		If Wheel Drives.	If Pinion Drives.	If Wheel Drives.	If Pinion Drives.
Arc of recess = $\frac{3}{4}$ pitch.	2	impossible	impossible	impossible	7
	3	"	"	"	41
	4	"	34	"	rack
	5	"	19	12	
	6	"	14	65	
	7	33	12	rack	
	8	16	11		
	9	11	10		
	10	9	10		

EPICYCLOIDAL GEARING (CONCLUDED).

	Number of Teeth in Given Pinion.	Least Number of Teeth in Spur Wheel		Greatest Number of Teeth in Annular Wheel	
		If Wheel Drives.	If Pinion Drives.	If Wheel Drives.	If Pinion Drives.
Arc of recess = ⅔ pitch.	2	impossible	impossible	impossible	11
	3	"	37	"	rack
	4	"	15	8	
	5	"	11	53	
	6	21	10	rack	
	7	11	9		
	8	8	8		

135. Unsymmetrical Teeth. — In all the figures of teeth hitherto given, the teeth are symmetrical, so that they will act whether the wheels be turned one way or the other. If the machine be of such a nature that the wheels are to be required to turn in one direction only, the strength of the teeth may be greatly increased by an alteration in form first suggested by Professor Willis. In Fig. 94 are represented two wheels, of which the lower is the driver, and always moves in the direction of the arrow. The describing circles are made large, thus reducing the obliquity of action. The right side of the driver's teeth and the left side of the follower's teeth are the only portions that are ever called into action; and they are made precisely as usual in the epicycloidal system. If the other sides were made the same, this would give a very weak form at the root. To obviate this, the back of each tooth is bounded by an arc of an involute. The bases of these involutes being proportional to the pitch circles, they will during the motion be sure to clear each other, because, geometrically speaking, they would, if the wheels moved the other way, work together correctly, though the inclination of their common normal to the line of centres is too great for the transmission of pressure. The effect of this shape is to produce a very strong form of tooth by taking away matter from the extremity of the tooth where the ordinary form has more than is required for strength, and adding it to the root.

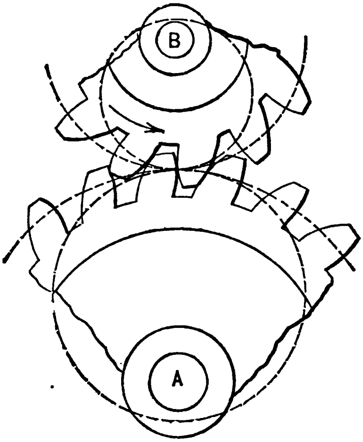


Fig. 94.

136. Twisted Gearing.—In this class of gearing (Art. 58) the point of contact travels, during the motion of the wheels, from one side to the other. The outer planes of the wheel should be twisted through an angle equal to the pitch, so that a fresh contact is always beginning on one side as the last contact is quitting on the other. In the double wheel shown in Fig. 39, there are, of course, two points of contact, travelling in a symmetrical manner with respect to the mid-plane of the wheel. The teeth must be so formed, that, when the angular velocity ratio is constant, contact shall only take place at the instant of crossing the line of centres. Otherwise, if the teeth were formed upon the usual principles, it is evident that the sliding contact before and after the line of centres would still remain. This may evidently be accomplished by making the flanks by any of the usual methods, and then making the faces so that they will lie *within* the faces which would be proper for a spur wheel with the flanks assumed. The simplest mode of making such teeth is to give them radial flanks, and make the faces semicircles whose diameter is the thickness of the tooth at the pitch circle. The motion is now transmitted by pure rolling contact, and the action of these wheels is exceedingly smooth and noiseless. They are, however, better suited for light work, because the pressure is confined to a single point, instead of being distributed along a line. For heavy work it is preferable to employ the stepped wheels (Fig. 37) in which the teeth are of the usual forms for spur wheels. In this case, the motion is, of course, no longer transmitted by pure rolling contact; but the action is, nevertheless, much smoother than that of ordinary spur wheels.

137. Non-circular Wheels.—In all the preceding cases of toothed wheels the pitch curves of the wheels have been circles; but the teeth may be just as well laid out when the pitch curves are not circular, though in the latter case the operation is much more tedious.

The two pitch curves must, in any case, be capable of *rolling* together with a constant velocity ratio. For instance, let it be required to lay out the teeth of a pair of equal ellipses. Divide the perimeter of the ellipse for the location of the teeth and the spaces. Find, by trial and error, the centre of curvature of the ellipse at the point where it is desired to draw a tooth outline. The tooth outline may then be drawn by rolling within and without the pitch ellipse a describing circle in the usual manner; the actual operation being performed by substituting for the pitch ellipse a circle whose radius is the radius of curvature of the ellipse at the point considered. By repeating this operation at successive pitch points, we can thus draw all the teeth. This method is perfectly general, and may be applied to rolling curves of any form, such as, for instance, the lobed wheels shown in Figs. 43 to 46. If the same describing circle be used throughout, its diameter should be such as to give radial flanks to the teeth in that part of the pitch line where the curvature is sharpest. Should other parts be very much flatter, the flanks of the teeth may spread too rapidly. This may be remedied by using different describing circles for the teeth in those parts, care being taken that the same one be always used for the face and flank that are to work together.

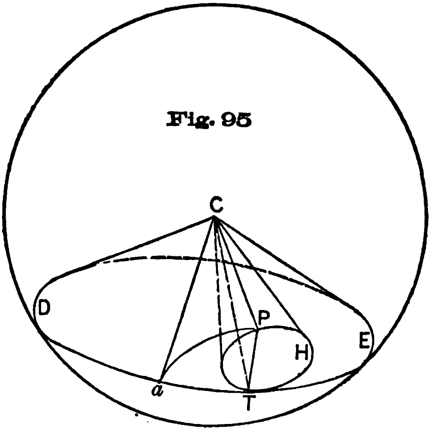
If one of the wheels be made a pin wheel, its pitch curve is to be used as the describing curve to generate the teeth of the other.

138. Bevel Wheels. — In all the cases of wheels previously considered, the pitch surfaces have been cylinders, all the transverse sections being consequently alike. Hence it was found most convenient to deal with one such section, so that the problems involved only lines instead of surfaces. But the pitch and describing curves employed, as well as the tooth outlines constructed, are merely transverse sections of surfaces whose elements are parallel to the axis of the wheel. Considering the cylinder as the special case of the cone in which the vertex is removed to an infinite distance, it would seem, that, in the case of the cone, the elements of the analogous surfaces should converge to the vertex of the cone.

In other words, just as we roll a *describing cylinder* within and without a *pitch cylinder* to generate the tooth surfaces of spur wheels, so may we roll a *describing cone* within and without a *pitch cone* to generate the tooth surfaces of bevel wheels. In both cases the line of contact of the tooth surfaces will be a right line; in the former it will be parallel to the axis of the cylinder, and in the latter it will pass through the vertex of the cone.

139. In Fig. 95, let *CDTE* be the pitch cone, and *CPTH* the describing cone; the two cones having the common vertex *C*, and being in contact along the right line *CT*. Draw any element, such as *CP*, of the describing cone, and consider the latter to roll to the left, keeping its vertex at *C*, and remaining always in contact with the pitch cone. *CT* is at any moment the instantaneous axis about which the plane *CPT* revolves; hence the surface *CPa*, generated by the line *CP*, will be normal to the plane *CPT*.

Fig. 95



We have seen that, with parallel axes, pitch curves may be selected which will produce a variable velocity ratio. Similarly, in bevel wheels, the bases of the cones might be so shaped as to produce changes in the velocity ratio. In practice, however, this is never done; any desired variation of velocity ratio being produced by some other means. We may, therefore, confine our attention to the case in which all the cones have circular bases. In this case, the point *P*, being at a constant distance from *C*, will move in the surface of a sphere of which *C* is the centre, and whose radius is equal to the slant height of the cones. The arc *TP* = arc

Ta; and the curve *Pa*, described by the point *P*, is a *spherical epicycloid.* Similarly, by rolling the describing cone within a pitch cone, a *spherical hypocycloid* will be generated. Following out the analogy between cylinder and cones, it is evident that, just as the tooth surfaces of cylindrical wheels are formed by moving a right line along the epicycloid and hypocycloid previously discussed, keeping the line always parallel to the axis of the pitch cylinder, so the tooth surfaces of conical wheels must be formed by moving a right line along the spherical epicycloid and hypocycloid, making the line in this case always pass through the common vertex of the pitch cones.

140. Construction of Tooth Outline. — The portion of the spherical surface occupied by the spherical epicycloid and hypocycloid, when they are used in the formation of teeth, is a narrow zone extending a short distance on both sides of the base circle of the pitch cone. For all practical purposes we may substitute for this narrow spherical zone a portion of the surface of a cone which is tangent to the sphere in the base circle of the pitch cone, and whose elements are consequently perpendicular to the corresponding elements of the pitch cone.

In Fig. 96, let *CA* and *CB* be the given axes of the pitch cones. Dividing the angle *ACB* so as to obtain the required velocity ratio, which in this case is $\frac{a'}{a} = \frac{3}{2}$, we find *CP*, the common element. The bases *FGP* and *EHP* are evidently small circles of the sphere whose radius is *CP*. Draw *PA* perpendicular to *CP*, and revolve it around the axis *CA*, generating the normal cone *FPA*. Similarly, draw *PB* perpendicular to *CP*, and revolve it about the axis *CB*, generating the normal cone *PBE*.

These new cones comply with the conditions above mentioned, and a narrow zone of their curved surfaces may be used upon which to describe the tooth outlines.

If, now, we roll a describing cone without one of the pitch cones and within the other, we will generate the tooth surface for the faces of the former and for the flanks of the latter. In order to construct this surface, we must select some particular element of the describing cone, and find the curve which it describes on the surfaces of the normal cones. To do this, we need only draw this element in successive positions, and find the points in which it pierces the normal cones. The curve formed by joining these successive points will be the directrix of the tooth surface; and the latter will be formed by moving a straight line along this generatrix, the line always passing through the common vertex of the pitch cones.

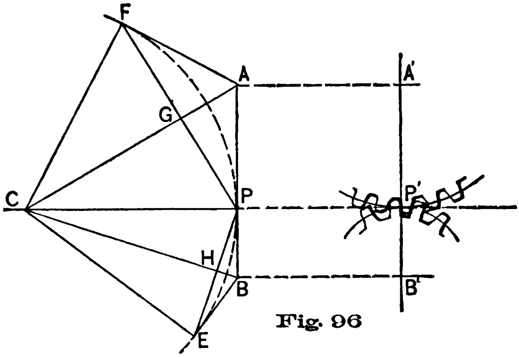


Fig. 96

This method will give the exact curves; the error of using the surface of the normal cone, instead of that of the sphere, being so small as to be inappreciable. Its application to practical cases involves more labor, however, than that of the following approximate method, which is the one in almost universal use.

141. Tredgold's Method. — If we assume the curved surface of each of the normal cones to be cut along one of the elements, and spread out on a plane, we will have (Fig. 96) portions of two circles whose radii, $A'P'$ and $B'P'$, are the slant heights of the cones. If, now, these circles be taken as pitch circles, and teeth be constructed on them by any of the usual methods for spur wheels, we may then wrap these surfaces, with the teeth, back into their original conical shape; and using the tooth curves, as they then appear on the normal cones, as directrices, we may generate the required tooth surfaces by moving a right line in contact with the curves, and passing through the common vertex of the cones, as before.

142. The practical method of drawing such teeth is shown in Fig. 97. Let *AC* be the axis of the bevel wheel, let *CDE* be the pitch cone, and *AED* the normal cone; *DNE* being the circular base common to both cones.

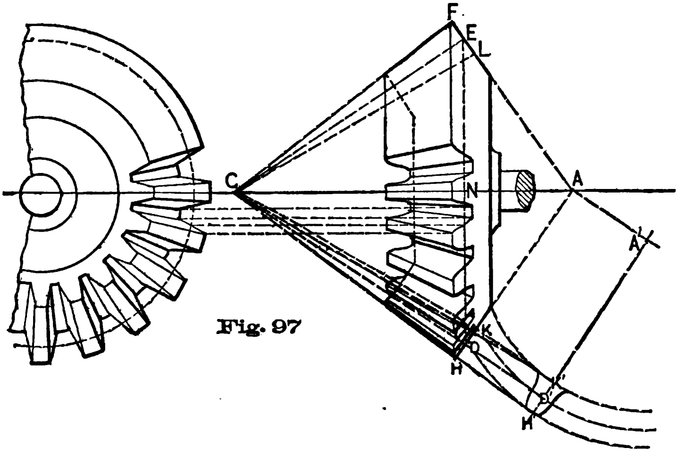


Fig. 97

In the side view, draw a line parallel to *AD*, and project the latter on it at *A′D′*. With centre *A′* and radius *A′D′*, describe a circular arc which will be an arc of the pitch circle to be used. On this arc lay off a tooth by the usual method, being careful to make the pitch an aliquot part of the circumference of a circle whose radius is *ND*. The tooth outlines may then be drawn by means of describing circles or by the approximate odontograph methods, according to the degree of accuracy required. Project $A'K'$, the radius of the root circle, at AK, and $A'H'$, the radius of the addendum circle, at AH. The points H, D, K, of the line AH will, by revolution, describe circles about AC, which will be represented in the side view by the straight lines FH, ED, and LK, and which will be seen in their true size and shape in the end view.

On the end view of the circles just mentioned we must next lay off, on each side of a radius, the half-thickness of the tooth at the top, at the pitch line, and at the bottom, as obtained from the development. If great accuracy is required, any number of additional circles may be used in a similar manner. Having thus determined the end view of the tooth outlines, we must next project each one to the side view; the points lying in each circle being projected to the straight line which is the side view of that circle. In practice, only frusta of conical wheels are employed, and the teeth are limited at both ends by normal cones. It is evident that in this case the shape of the teeth will be similar at both ends, except that the outer ones will be larger in proportion to their greater distance from the vertex. The points of the inner tooth outlines are found by drawing radii through the principal points of the outer tooth outlines already determined, and finding the intersection of these radii with the circles corresponding to the inner normal cone.

It may be required to describe the teeth by either the epicycloidal or the involute system, or so that they may be used for an annular bevel wheel; but the modification of the general operation is in each case similar to the corresponding modification for wheels on parallel axes.

143. Relative Action of Bevel and Spur Wheels. — The action of a toothed wheel, other things being equal, is always more smooth in proportion as the teeth increase in number and decrease in size, because these conditions diminish the obliquity of action, as well as the amount of sliding. But in bevel wheels the action of the outer tooth outlines does not deviate much from the plane tangent to the two normal cones at P (Fig. 96), and hence they act the same as spur wheels having the radii AP, BP, which are larger than the radii of the bevel wheels themselves in the ratios $\frac{AP}{GP}$ and $\frac{BP}{HP}$. But $\frac{AP}{GP} = \frac{CP}{CG}$, and $\frac{BP}{HP} = \frac{CP}{CH}$. In other words, the action of a bevel wheel, so far as it is affected by the number of its teeth, is equal to that of a spur wheel of the same pitch whose radius is greater than that of the given bevel wheel in the same ratio that the slant height of the pitch cone is greater than its altitude.

In a pair of mitre wheels this ratio is $1\frac{4}{10}$, so that the action of a mitre wheel having, say, fifty teeth is equivalent to that of a spur wheel of seventy teeth.

144. Skew Bevel Wheels. — The theoretical construction, as well as the practical manufacture, of the *exact* forms of teeth for skew bevel wheels, are both extremely complicated and laborious operations, and consequently they are rarely employed in practice. When skew bevel wheels are to be used, however, their teeth may be laid out by the following *approximate* method, which will give results abundantly accurate for all practical purposes. Having determined the pitch surfaces, as in Fig. 26, and decided on the frusta to be employed, we draw, at each end of each frustum, a cone normal to the respective hyperboloid. These cones are then to be developed, and teeth are to be laid out on them according to Tredgold's method. In this construction, it must be borne in mind that the relative numbers of teeth of the two wheels are not in the same proportion as the radii of the base circles, as in the case of cones; for (Eq. 7, p. 40) these numbers are evidently proportional to the sines of the angles made by the projection of the common element with the projections of the respective axes, these projections being made on a plane parallel to the common element and both axes. The two wheels, then, have different circumferential pitches, and the ratio of the latter must be determined in each special case. The teeth having been laid out on the development of the normal cones, the latter are then to be replaced, the outer and inner cones being given the proper position with regard to each other by bringing the pitch points of a pair of corresponding tooth curves on the same generatrix. The surfaces of the teeth will be formed by joining the corresponding points of these curves by right lines.

145. As an illustration of the method of Art. 47, let the axes be given as in Fig. 26, and let it be required to connect these axes so that $\frac{a'}{a} = 3$. Let DB' be taken as the intermediate axis, and let a'' represent its angular velocity. Dividing the fraction $\frac{a'}{a}$ into two factors, such as $\frac{a''}{a} = \frac{3}{2}$ and $\frac{a'}{a''} = 2$, we have simply to solve two ordinary bevel wheel problems: first, to connect the axes DR' and DB' so that $\frac{a''}{a} = \frac{3}{2}$; second, to connect the axes DB' and $B'O'$ so that $\frac{a'}{a''} = 2$. As DB' is perpendicular to both axes, the angles to be divided according to the method of Art. 38 are both right angles, so that the construction is very simple.

If the axes pass so near each other that the common perpendicular is too short to be used, some other line, such as RO, must be taken. In this case, it becomes necessary to determine the true size of the angles PRO and POR; and this is most conveniently done by revolving the line RO in turn about each of the axes until it is parallel to the vertical plane of projection, when the angle which it makes with the respective axis will be shown in its true size.

146. Axes neither Parallel nor Meeting connected by Wheels with Involute Teeth. — In Fig. 74 we have shown a pair of wheels with involute teeth, *DE* being the line of action. In the figure the wheels are in the same plane, and the point of contact is always situated in the line *DE*.

The upper wheel remaining fixed, suppose the plane of the lower wheel to be revolved through any given angle about the line *DE*, as on a hinge. The two wheels will now lie in different planes, their axes being neither parallel nor intersecting. The line *DE* will be the intersection of these two planes; and the position of each wheel in its own plane, with reference to that line, is unaltered. But *DE* is the locus of contact; and, as the position of neither wheel with reference to *DE* has been changed, it follows that the velocity ratio of the wheels will not be affected by the inclination of their planes. When the wheels are so inclined, they can, of course, move only in the direction which makes *DE* the locus of contact. If they are required to move in the reverse direction, they must be swung about a line similarly inclined to the line of centres in the opposite direction; but it is evident that in no case can they drive in both directions except when they are in the same plane.

This property of involute teeth, of transmitting motion between axes neither parallel nor meeting, is only true when the wheels are very thin; so that in practice the teeth of one wheel must be rounded so as to touch those of the other in *points* only, and not in *lines*.

147. Face Gearing. — Before the introduction of bevel gearing, the problem of transmitting motion between axes that were not parallel was usually solved by means of face gearing. Let two face wheels with cylindrical pins, exactly alike in every respect, be placed in gear, as shown in plan and elevation in Fig. 98, with their axes at right angles; the latter not meeting in a point, but having their common perpendicular equal to the diameter of the pins. Then will these wheels revolve together with the same angular velocity.

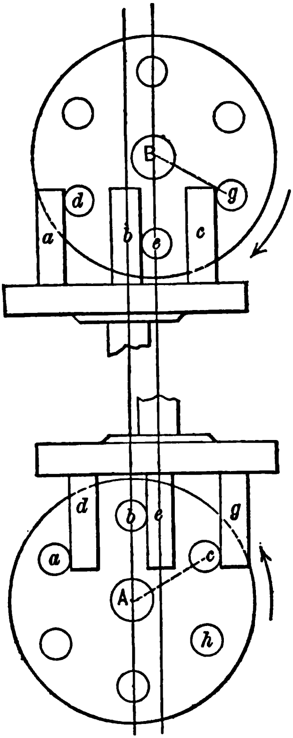


Fig. 98

Let B be the driver, and let the pins c, g, be in contact. The distance between the axes of these pins is the sum of the radii of the pins; that is, the diameter of a pin, or, what is equal to this diameter, the perpendicular distance between the axes of the wheels.

Let the driver B turn, in the direction of the arrow, through one-sixth of a revolution; the pin g moving to the position *e*, and driving before it the pin *c* to the position *b*. The distance between the axes of the pins is equal to the diameter of a pin, as before; and consequently the length of the perpendicular let fall from *g* on *Be* must equal the length of the perpendicular let fall from *c* on *Ab*. In other words, $Bg \sin gBe = Ac \sin bAc$; and, as $Bg = Ac$, we have $\sin gBe = \sin bAc$. Hence angle $gBe =$ angle bAc, which proves the equality of the angular velocities.

The driver was in this case supposed to turn through an angle of sixty degrees; but this was merely a matter of convenience, as the same proof could have been applied to any other angle. The pin *g* must not be so long that its end will come into contact with the pin *h*, as the wheels revolve in the directions of the arrows. This consideration fixes the maximum length of the pins, which is the same in both wheels.

148. Axes Intersecting.—As the common perpendicular to the two axes becomes less, the diameter of the pins decreases; so that, when the axes intersect, the pins become mere lines. In order to transmit any power, the pins must manifestly have some thickness; but they cannot be cylindrical on both wheels. The pins on one wheel may, however, still be cylinders, in which case the shape of those on the other may be found in the following manner. Suppose the axes of the wheels shown in Fig. 98 to be brought together so as to intersect, the pins thus reducing to mere lines. Instead of having the corresponding pins in contact, as in Fig. 98, let the lower wheel be turned through a small angle, so as to separate the pins by some arbitrary distance, as shown in Fig. 99. Now, if both wheels be turned, in the directions of the arrows, with the same angular velocity, it is evident that the common perpendicular between any two corresponding pins will change according to the positions of the pins at any instant; and the length of this perpendicular can readily be determined for any positions of the pins. If, still using mere lines for the pins of the upper wheel, we now expand the pins of the lower wheel into solids of revolution, the radius of whose cross-section shall at any height be equal to this common perpendicular, it is evident that the two wheels will work together with a constant velocity ratio.

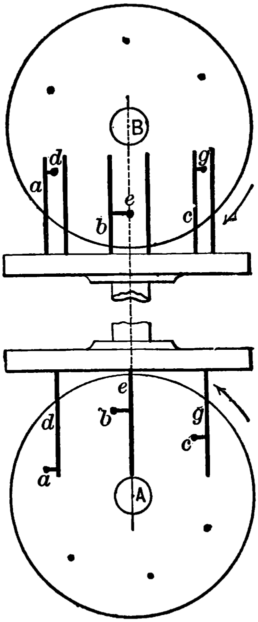


Fig. 99

If we now expand the mere lines, which act as pins of the upper wheel, into practical cylinders, their projections will become circles, and the curves for the lower pins will then be found by a process similar to that employed (Art. 122) in pin gearing.

The principal advantage of face gearing is the facility of turning the pins and cogs in a lathe; but on the other hand, we have the serious drawback, that the pressure between the pins is exerted at a single point only.

149. Intermittent Gearing. — In Fig. 100 is shown the manner of connecting a pair of axes so that a uniform rotation of the one shall produce a given intermittent motion of the other. In such motions, the maintenance of an *exact* velocity ratio is not usually essential, the important points being that the follower shall be caused to turn through a *definite* angle, and then be securely held in its new position until it is again to be put in motion.

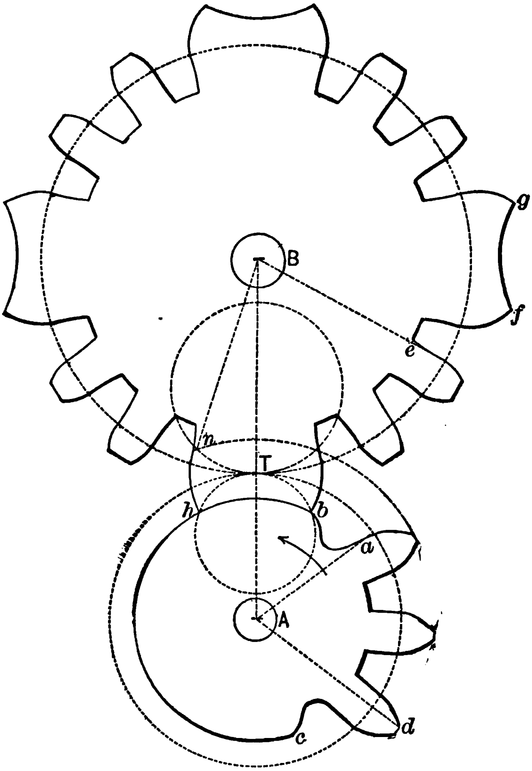


Fig. 100

Suppose the axes A and B to be given in position, and let it be required to connect them so that during every revolution of the driver, the follower shall turn through ninety degrees, the velocity ratio during the motion of the latter being, as near as may be, $\frac{a'}{a} = \frac{2}{3}$. The diagram shows the solution of this problem, the pitch circles of the two wheels being drawn of the proper size to give the velocity ratio, as usual. The teeth are formed in the usual manner by means of the describing circles shown. The projections on the follower have the same outlines as the other teeth, except that they are longer, and their tops are connected by an arc of the same radius as the smooth portion of the driver.

Supposing the driver to move from its present position in the direction of the arrow, the smooth arc of the driver will slide along that of the follower, and no motion will be produced in the latter until the point b of the driver reaches the line of centres AB. At this moment the point a of the driver's tooth will come into contact with the point b of the follower, and the latter will begin to move. The wheels will then continue in gear, the different teeth coming successively into action, and motion being transmitted with the exact velocity ratio, $\frac{a'}{a} = \frac{2}{3}$, until contact ceases between the point d of the driver and the point e of the follower. These two points, at the moment of quitting contact, will be at the point n of the upper describing circle. As soon as these points have quit contact, the edge c of the driver will operate on the edge fg of the follower, turning the latter into the position now occupied by the edge bh; and it will be held in that position until the point a again comes round into contact with g. The velocity ratio in this motion is exact while the driver moves through an angle aAd, and the follower through an angle eBn.

CHAPTER IX.

COMMUNICATION OF MOTION BY SLIDING CONTACT.

VELOCITY RATIO AND DIRECTIONAL RELATION CONSTANT OR VARYING.

Cams. — Endless Screw. — Slotted Link. — Whitworth's Quick Return Motion. — Oldham's Coupling. — Escapements.

150. In the last four chapters have been discussed the cases of sliding contact where both the velocity ratio and the directional relation were necessarily constant; in the present chapter will be presented the various arrangements in which either or both of these may vary.

151. A Cam is a plate which transmits motion to its follower by means of its curved edge, or by means of a curved groove cut in the surface of the plate. When the motion is small or intermittent, such plates are often called *tappets*, or *wipers*.

In most cases which occur in practice, the conditions to be fulfilled in designing a cam or wiper do not directly involve the velocity ratio; usually a certain series of definite positions is assigned which the follower is to assume when the driver is in a corresponding series of definite positions. In cam motions, the motion of the follower is usually derived from the cam by means of a cylindrical roller turning about a smaller pin as an axis, the latter being rigidly fastened to the follower. This has the advantage that nearly all the wear is concentrated on this axis, which may readily be renewed when worn out. If the pin is to be driven by the cam in one direction only, being made to return by the force of gravity or the elastic force of a spring, the cam need only have one acting edge; but if the cam is to drive the pin in both directions, it must have two acting edges, with the pin between them, so as to form a groove or a slot of a uniform width equal to the diameter of the pin, with clearance just sufficient to prevent jamming or undue friction.

The centre of the pin may be treated as practically at all times coinciding with the centre line of such a groove, which centre line may be called the pitch line of the cam.

The most convenient way to design a cam is usually to draw in the first place its pitch line, and then to find the acting edge or edges by the process of Art. 122; using a radius slightly greater than that of the pin in case two edges are employed.

152. Construction of the Cam Curve. — In Fig. 101, let A be the centre of motion of the proposed cam, and BH the path of the centre of the pin on the follower; the condition being that the centre of the pin shall start from the point H, and assume in succession the positions E, D, C, and B while the cam revolves through successive angles of thirty degrees. With centre A and radius AH, describe the circle HN; produce the radius AH to S, and draw the other radii (produced), AK, AL, AM, and AN, at successive angular intervals of thirty degrees. With centre A, draw circular arcs through the successive positions E, D, C, B, of the pin, and on these arcs lay off the distances $Kk = Cc$, $Ll = Dd$, $Mm = Ee$. Then will k, l, and m be points of the cam curve required. The curve $nmlkB$, drawn through these points and N and B, will be the curve which will fulfil the required conditions; for, assuming n to be at H, and the cam to revolve in the direction of the arrow, it is evident that as the radii AM, AL, AK, and Ab successively come into the position AS, the joints m, l, k, and B of the cam curve will coincide with *E*, *D*, *C*, and *B* respectively, thus driving the pin as required. To find the curve for a pin of sensible diameter, we proceed as in Art. 122, drawing circles of the same diameter as the pin in a sufficient number of positions along the pitch line already found, and then drawing the acting edge tangent to these circles.

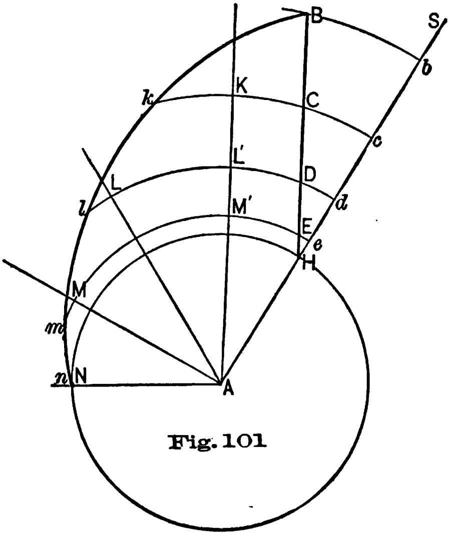


Fig. 101

When the path of the pin passes through the centre of motion of the cam, the distances *Ee*, *Dd*, etc., all reduce to zero; and the pitch line is drawn through the points of intersection of the successive radii and the circular arcs through the corresponding positions of the pin.

As the angle *BHS* increases, the action between the edge of the cam and the pin becomes more oblique, thus increasing the friction; and it is hence advisable to make that angle as small as possible; in other words, the path of the pin should point as near as possible to the centre of motion of the cam.

In case the motion of the follower is required to be uniform, the distances *HE*, *ED*, *DC*, and *CB* would all be equal, but no modification of the method of construction would thereby be introduced.

153. Another Example.—In Fig. 101 the path of the follower is a straight line, and the cam has uniform motion about a fixed centre. But none of these conditions need be adhered to. The path of the follower may be any curve whatever, and it may move in this path in either direction, and with uniform or varying velocity. The cam usually revolves about a centre, or has rectilinear motion; but its velocity may also be varied at pleasure. All these possible variations give rise to an endless variety of shapes for the cam curves, but the principles underlying their construction are always the same. Thus, in Fig. 102, let the path of

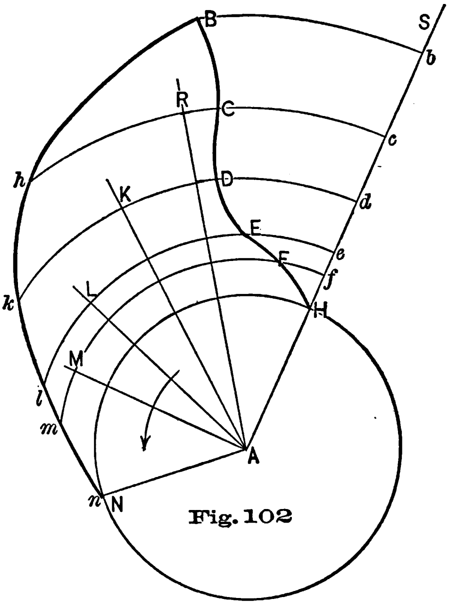


Fig. 102

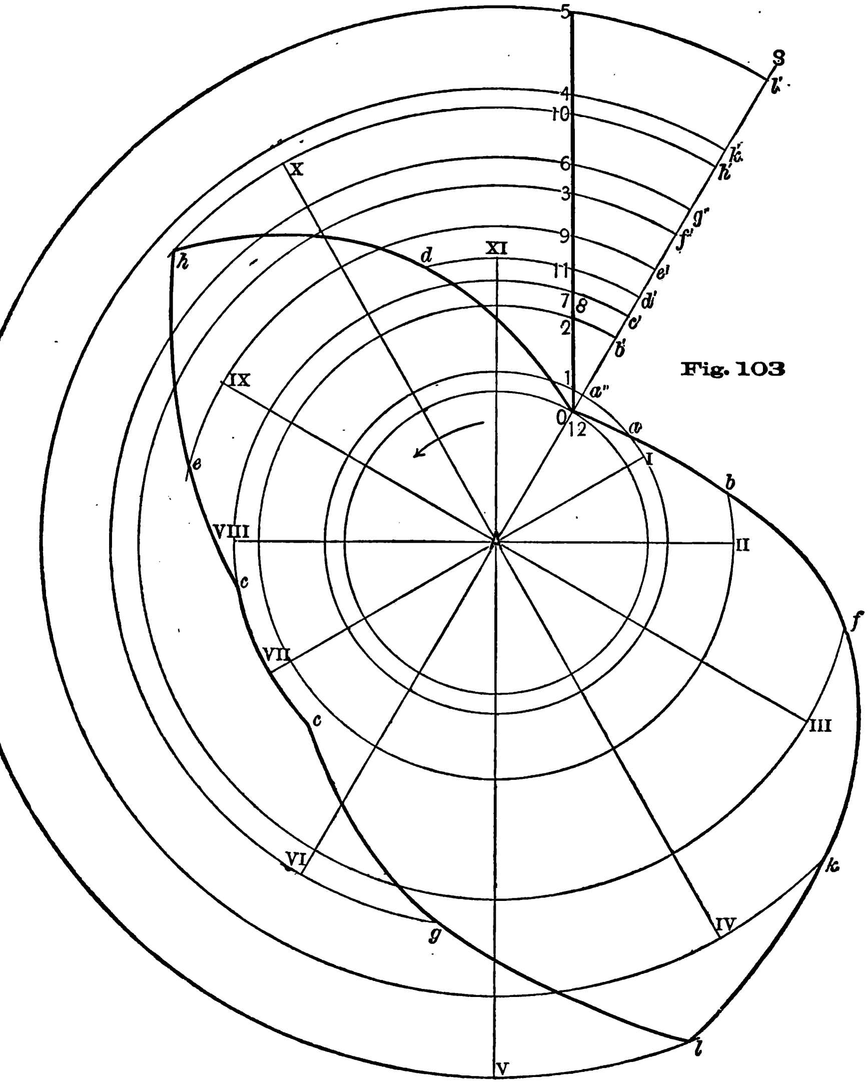
the pin be the curved line *HB*, and let the pin successively occupy the positions *F*, *E*, *D*, etc., while the cam revolves in the direction of the arrow through the unequal angles *NAM*, *MAL*, *LAK*, etc. The radii being drawn at the given angles, circular arcs are drawn through *F*, *E*, *D*, etc., and the points of the curve found, just as in Art. 152, by making $Rh = Cc$, $Kk = Dd$, etc.

154. Cam for Complete Revolutions. — In Figs. 101 and 102 the directional relation is constant; in other words, the direction of rotation of the cam must be reversed in order to bring the pin down again to *H*. But this may be accomplished by simply adding to the curve of the cam, in which case the latter may revolve continually in the same direction. The law of motion of the pin in one direction may be entirely different from that in the other direction, and the pin may be given an interval of rest at any elevation by making the corresponding part of the cam curve an arc of a circle.

In Fig. 103, let *A* be the centre of motion of the cam, and let the vertical numbered line be the path of the follower. The cam is to revolve uniformly at the rate of one revolution in twelve seconds.

Each number on the vertical line shows the required position of the pin at the end of the second indicated by that number.

Draw twelve equidistant radii, and draw circular arcs through the various positions of the follower. Making $Ia = 1a'$, $IIb = 2b'$, $IIIf = 3f'$, $IVk = 4k'$, etc., we find the points of the curve, as before. The interval of rest indicated by the coincidence of the numbers 7 and 8 is obtained by means of the circular arc *cc*. The cam in the figure is drawn in the position when the pin is at the point *O* 12; and the cam is ready, by one complete revolution in the direction of the arrow, to cause the pin to go through the cycle of motion required.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l'
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3
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XI
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12
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I
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b
VIII
II
c
A
f
VII
c
III
VI
k
g
IV
l
V

Fig. 103

155. Cam moving in Straight Path.—In all the preceding cases we have assumed the cam to revolve about some fixed centre of motion. But this is not a necessary condition; it may move in any path whatever. In practice, however, there is but one other path employed; viz., the straight line. In Fig. 104, let *ABCD* be a flat rectangular plate moving in the direction of its length, and let the path of the follower be the line *MN*, at right angles with the direction of motion of the plate. Let the pin of the follower start from a position of rest at *P*, and move with a gradually accelerated velocity, so that it occupies the positions 1, 2, 3, 4, etc., at the end of equal successive intervals of time. Lay off on *AB* the distances *MI*, *I II*, *II III*, *III IV*, etc., through which the plate moves during the same intervals. In the figure the plate is supposed to be moving uniformly, and these distances are consequently equal; but they may vary according to any assigned law. Draw the lines *Ia*, *IIb*, *IIIc*, *IVd*, etc., parallel to *MN*, and the lines 1*a*, 2*b*, 3*c*, 4*d*, etc., parallel to *AB*. Their intersections, *a*, *b*, *c*, *d*, etc., will be points of the required cam curve. From this the working curves are derived in the usual manner, being the

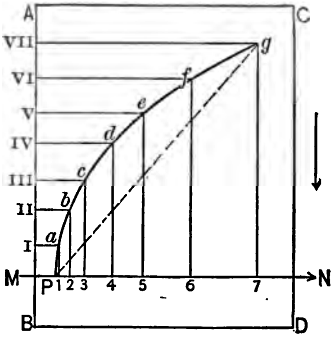

Fig. 104.

sides of a groove or slit of the proper width in the plate. The theoretical curve above found is the centre line of the groove.

In the special case shown in the figure, the lines *Ia*, *1a*, *IIb*, *2b*, etc., are at right angles; the angle between them is always the angle between the directions of motion of the plate and the follower. If in Fig. 104 we make the velocity of the follower also constant, the cam curve will become a straight line; for instance, if we assume that the follower is to traverse the distance *P7*, with a uniform velocity, during the same time that the plate moves, also with a uniform velocity, over the distance $M\ VII = 7g$, then the straight line *PG* must be the cam line required. This line will evidently be the hypothenuse of a right triangle, the other two sides of which are the lines representing the respective distances traversed by the plate and the follower in the same time. The velocity ratio of the cam plate and the follower in this special case is evidently constant, and is simply the ratio of the isochronous distances above mentioned.

156. The Screw. — If a plate with a straight slit or groove, as just described, be wrapped around a cylinder whose axis is parallel to the path of the follower, the slit in the plate will become a spiral groove in the cylinder. If the cylinder be revolved uniformly, this groove will impart precisely the same motion to the follower as before.

If the length of the plate be greater than the circumference of the cylinder, the spiral groove will encompass its surface through more than one convolution, and may in this way proceed in many convolutions from one extremity of the cylinder to the other. Such a recurring spiral is called a *screw*. The inclination of the spiral to a line drawn on the surface of the cylinder parallel to the axis is constant, and is the same as the inclination of the straight line in the flat plate to the path of the follower.

The *pitch* of a screw is the distance between successive convolutions of the spiral measured along a rectilinear element of the cylindrical surface.

The screw is sometimes made in this elementary form, consisting of a simple spiral groove which gives motion to a slide, by means of a pin fixed to the latter, and lying in the groove; but generally screws receive a more complex arrangement.

In the first place the pitch is made comparatively small, the necessary motion of the follower being secured by a corresponding increase in the number of revolutions of the screw. The convolutions of the groove are brought so close together that the ridge which separates two contiguous grooves becomes the counterpart of the groove itself. This ridge is termed the thread of the screw; and from its section the screw derives its distinctive title, such as square-threaded, V-threaded, and round-threaded.

In the second place, instead of a single pin, other pins may be fixed to the follower opposite the other convolutions; then, since each pin will receive an equal velocity from the revolving cylinder, the motion of the follower will be effected as before, with the advantage of an increased number of points of contact. But this series of pins may be replaced by a short comb or rack, the outline of which exactly fits that of the threads of the screw. This is the most ancient form in which the screw was employed.

Most commonly, however, the piece which receives the action of the screw is provided with a cavity embracing the screw, and fitting its thread completely; being, in fact, a hollow screw corresponding in every respect to the solid screw. Such a piece is termed a *nut*, and the hollow screw an *inside screw*, the solid screw being then called an *outside screw*. These modifications are only introduced to distribute the pressure of the screw upon a greater surface; for, as the action of the thread is exactly alike upon every section of the nut, the result of all these conspiring actions is the same;

namely, that the piece to which the pin or comb or nut is attached advances in a direction parallel to the axis of the screw through a distance equal to the pitch for every revolution.

157. A screw may be *right-handed* or *left-handed;* the majority of screws are the former, the latter being used only when other conditions make it necessary. Supposing the nut to be fixed, a *right-handed* screw will enter its nut when turned in the direction of the hands of a clock; a *left-handed* screw must be turned in the opposite direction.

If the inclination of the thread of a screw to the rectilinear elements of the cylinder be very great, one or more intermediate threads may be added. In such cases the screw is said to be *double-threaded*, *triple-threaded*, etc., according to the number of separate spiral threads on the cylinder.

Screws whose pitch is an aliquot part of an inch are usually classified by mentioning the reciprocal of the pitch; i.e., if a certain screw has a pitch of one-quarter of an inch, it is spoken of as having four threads to the inch.

During one complete revolution of any screw, the follower will evidently move through a space equal to the pitch of the screw; i.e., through a space equal to the distance between successive convolutions of the *same* spiral measured on a rectilinear element of the screw cylinder.

When the comb or rack form above spoken of is used, the screw is frequently made short, and the rack lengthened. If it is essential that the screw shall always remain completely in gear with the rack, then the maximum length of path described by the latter will be the difference between their lengths.

158. Endless Screw. Worm and Wheel. — From the rack driven by a short screw, we readily pass to the so-called *endless screw*, shown in Fig. 105. In this combination the screw, or *worm*, *BB*, gives motion, not to a rack, but to the wheel *C*. The screw is so mounted that it can have no motion except that of rotation, and the wheel has teeth of the same pitch as the screw thread. If the screw axis be turned around, every revolution will cause one tooth of the wheel to pass across the line of centres; and as this action puts no limit, from the nature of the contrivance, to the number of revolutions in the same direction, a screw fitted up in this manner is termed an endless screw, in opposition to the ordinary screw, which, when turned around a certain number of times either way, terminates its own action by bringing the nut to the end of the thread.

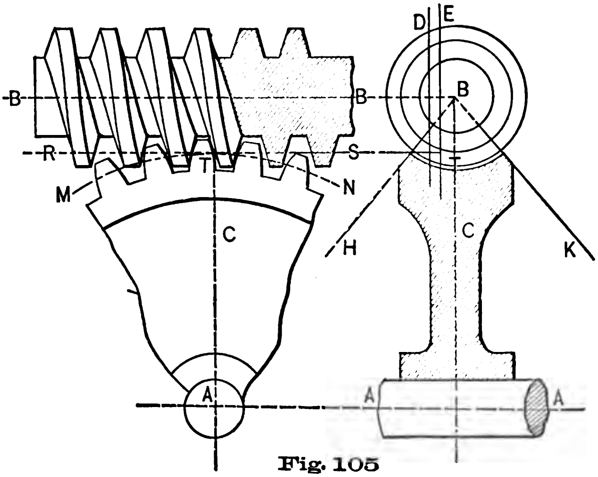

Fig. 105

159. Shape of the Teeth.—If we make any meridian section of the screw, we will find it to be a rack; in fact, the screw may be moved in the direction of its length, and will then drive the wheel precisely in the manner of a rack. Consequently, if the wheel be merely a thin plate, we need only make the meridian section of the screw a rack with teeth laid out in the usual manner to work correctly with the assumed wheel tooth. But as in practice the wheel must be given some thickness, it is necessary to determine the proper form of the teeth for that case.

If we make a series of sections through *D*, *E*, etc., parallel to the mid-plane of the wheel, we will still find the section of the screw to be a rack, though the outlines of the teeth will change in shape in each section.

If, now, we make the outline of any section of the wheel tooth of the proper shape to gear correctly with the outline of the corresponding section of the screw tooth, considered as the tooth of a rack, we shall evidently have a point of contact between the corresponding teeth in every section.

In fact, if we make a wheel tooth whose shape is continually changing in every section to correspond to the change in the same section of the screw tooth, we shall have the teeth in contact at each instant along a line, so that the wear will be distributed along a surface. Such a screw is called a close-fitting or tangent screw.

160. Practical Method of Cutting Wheel Teeth. — The practical difficulty of making the teeth of a wheel of which the form in every parallel section shall be different, is very simply overcome by making the screw cut the teeth.

An exact copy of the tangent screw is made of steel, the edges of its threads are notched, and it is then hardened, so that it becomes a cutting tool. It is then mounted in a suitable frame, so as to gear with the roughly formed teeth on the wheel, and turned so as to drive them; in the course of which operation it cuts them to the proper figure. The axis of the cutting screw is placed at first at a distance from the axis of the wheel somewhat greater than the intended permanent distance; and, after each complete revolution of the wheel, the axes are brought nearer together, until the permanent distance is attained; and, by turning the screw in this last position, the shaping of the teeth is finished. An involute wheel tooth working with a screw tooth whose meridian section has straight, sloping sides, is the best combination, as the successive diminutions of the distance between the axes will not affect the velocity ratio (Art. 108). In order to secure a good arc of action, and diminish obliquity, such wheels should not be given less than about thirty teeth.

In order to avoid weak corners in the wheel teeth, their sides are usually bounded by straight lines, *BH* and *BK*, radiating from the axis of the worm; and the angle *HBK* usually varies between sixty and ninety degrees.

161. Hour-Glass Worm. — Instead of making the pitch surface of the worm a cylinder, we may make it conform to the curvature of the wheel. In that case its pitch surface will be the surface produced by revolving an arc of the wheel pitch circle about the axis of the worm, thus forming the shape from which the worm derives its name. This arrangement is also named, after its inventor, Hindley's screw. The acting surfaces of both the worm and the wheel are very peculiar; but the arrangement may, nevertheless, be very easily constructed in practice.

Just as in the ordinary tangent screw, we must first prepare a cutting screw. To obtain this, a tool whose cutting edges are formed in the shape of the proposed wheel tooth is so clamped to a horizontal revolving plate of the size of the proposed wheel that the plane of its cutting edges passes through the axis of the worm. The plate and the worm blank being rotated at their proper relative velocities by means of some interposed mechanism, the distance between the two axes is gradually diminished until the permanent distance is reached, during which operation the worm will be cut to the proper shape. By taking such a worm, notching its edge to make a cutting tool of it, the wheel teeth can then be cut just as in the ordinary worm and wheel.

Such teeth are in contact along a line in the meridian plane of the screw, but do not come in contact along the whole surface of a wheel tooth.

162. The endless screw falls under the case of two revolving pieces whose axes are not parallel and do not meet. It communicates motion very smoothly, and is equivalent to a wheel of a single tooth, because one revolution passes one tooth of the wheel across the line of centres; but, generally speaking, it can be employed only as a driver, on account of the great obliquity of its action. A worm may be multiple-threaded, just like any other form of screw, and, in that case, will pass as many wheel teeth across the line of centres for every revolution of the worm as there are separate threads on the latter. The practical process of cutting the teeth is, however, the same as before.

163. Screw to produce Variable Motion. — In all the cases previously described, the screw has been supposed to have a uniform pitch, and hence to produce a uniform motion in the follower. But we may impart any motion whatever; the only condition being, that the pitch of the follower must not deviate much from a straight line parallel to the axis of the screw. As the inclination of the spiral groove varies, the velocity of the follower changes; a period of rest of the follower being obtained by making the inclination zero. A small intermittent motion may readily be obtained by making the groove in the shape of a simple ring, except at a certain portion, where it deviates the necessary amount.

If it be required that the follower shall move back and forth while the screw revolves continually in the same direction, the spiral must be cut in both directions; in which case the follower cannot be a rack or nut or wheel, but must be a single pin or similar piece. On the cylinder of the screw are cut two complete spirals, one right-handed and the other left-handed, joined together at their ends; so that the two screws form one continuous path, winding around the cylinder from one end to the other and back again continuously. When the cylinder revolves, the piece which lies in this groove, and is attached to the follower, will be carried backwards and forwards; and each total oscillation will correspond to as many revolutions of the cylinder as there are convolutions in the compound screw. As the screw grooves necessarily cross each other, the piece that slides in them must be made long, so as to occupy a considerable length of the groove; thus making it impossible for it to quit one screw for the other at the crossing places. Also, as the inclinations of the two screws are in opposite directions, it is necessary to attach that piece to the follower by means of a pivot, so as to allow it to turn through a small arc as the inclination changes. By varying the inclination at different points, the velocity ratio may be varied at those points.

164. Pin and Slotted Crank.—In Fig. 106, let *A* be the centre of rotation of an arm, *AP*, carrying at its extremity a pin, *P*, which slides freely in the slot in the piece *BC*. The latter has its centre of rotation at *B*. If the arm *AP* be revolved uniformly, it will impart a variable velocity to the arm *BC*.

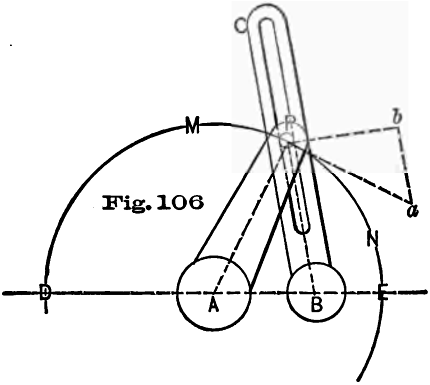

Fig. 106

Let *Pa*, perpendicular to *AP*, represent the *linear* velocity of the pin in the circle *MN*. Draw an indefinite line perpendicular to PB at P, and let fall on it the perpendicular ab; then will Pb be the linear velocity of the point P of the arm BC at that instant. Let a = angular velocity of the arm AP, and a' = angular velocity of the arm BC. Also let the constant length AP be designated by R, and the variable length BP by r.

Then

$$a = \frac{Pa}{R}, \quad a' = \frac{Pb}{r}.$$

But

$$Pb = Pa \cos \overset{a}{\cancel{A}}Pb = Pa \cos APB,$$

hence

$$a' = \frac{Pa \cos APB}{r}, \quad \text{and} \quad \frac{a'}{a} = \frac{R}{r} \cos APB.$$

When $APB = 0$, that is, when both the arms lie in the line of centres DE, the limiting values of the velocity ratio will be obtained. When P is at E, the velocity ratio $\frac{a'}{a}$ has its maximum value, $\frac{R}{r} = \frac{R}{R - AB}$. The ratio becomes smaller as P leaves E and approaches D, at which point $\frac{a'}{a}$ has its minimum value, $\frac{R}{r} = \frac{R}{R + AB}$.

So long as AB is less than R, we may by this means cause one arm to revolve with a variable velocity by means of another arm revolving uniformly.

When AB exceeds R, the second arm merely swings on each side of the line of centres through an angle whose sine is $\frac{R}{AB}$. When it is at the end of its outward swing, the angle $APB = 90°$ and $\frac{a'}{a} = \frac{R}{r} \times 0 = 0$; showing that for that instant no motion is imparted to the arm BC by the rotation of AP.

The necessary length of slot when *B* lies within the circle *MN* is the diameter of the pin + *BD* − *BE*; when *B* lies without *MN*, the necessary length of slot is the diameter of the pin + 2*AP*.

165. Whitworth's Quick Return Motion.—If in Fig. 106 we attach a connecting-rod to the end *C* of the arm *BC*, and compel the other end of the rod to move in a straight line perpendicular to *AB* at *B*, we will have a combination such as is represented by Fig. 107. The length of the stroke is evidently 2*BC*. If the arm *AP* be revolved uniformly, the forward and back strokes of *Q* will be made in times proportional to the arcs *adb* and *acb*. We thus have a form of "quick return" motion. This has been applied in a modified form, as shown in Fig. 108, to a shaping machine, by Sir J. Whitworth.

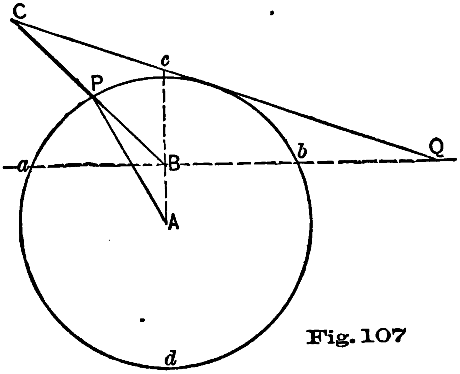

Fig. 107

In the figure, *D* is a plate spur wheel which turns about its centre *A*, upon the large fixed shaft. A pin, *P*, fixed in and projecting from the face of wheel *D*, corresponds to the point *P* in Fig. 107, so that *AP* is the arm which revolves uniformly. A pin, *B*, eccentric in the large shaft, is the centre about which the arm of varying length turns; *BP* corresponding to *BP* of Figs. 106 and 107. A crank piece, *E*, turns about *B*, and has a slot in one end, in which *P* slides. To the opposite end of this piece the connecting-rod is attached. The end *Q* of the rod is attached to the sliding head which carries the cutting tool.

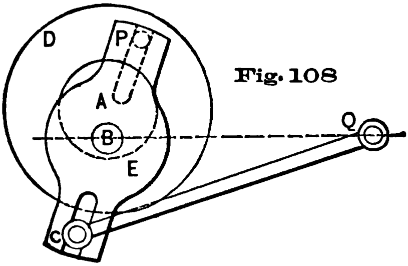

Fig. 108

As *D* revolves, motion is given to *Q* by means of the pin *P* and the crank piece, and the varying distance of *P* from *B* exactly replaces the arm of varying length. The length of stroke is adjusted by altering the position of *C* in that end of the crank piece, thus changing the length of the crank arm *BC*, but in no way affecting the ratio of the periods of advance and return. Thus, for example, if the arc *acb* (Fig. 107) is one-third of the circumference, *adb* being two-thirds, the period of advance is to the period of return as 2 is to 1, without regard to the actual length of stroke.

166. Pin and Slotted Sliding Bar.—In Fig. 109, let the pin *P* be fixed at the extremity of the uniformly revolving arm *AP*, as before. The piece *B* is free to move in the direction *CD* or *DC* only, and has in it a slot perpendicular to the line *DC*, in which the pin slides. Let $Pa = V =$ linear velocity of the pin in the arc of the circle; then v, equal to linear velocity of the piece *B* in the direction *AC*, will be found, as in the last article, by dropping the perpendicular *ab* on the line *Pb*, the latter being perpendicular to the line of the slot.

groove, and is attached to the
wards and forwards
spond to as many revolutions
convolutions in the
necessarily cross each other
must be made long, so as to
the groove; thus making it impossible
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164. Pin and Slot
the centre of rotation of an

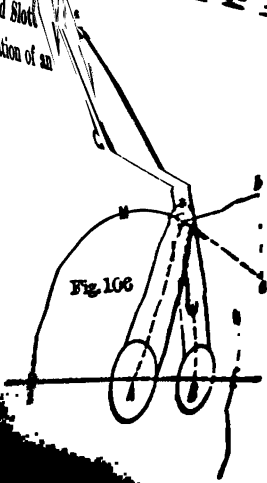
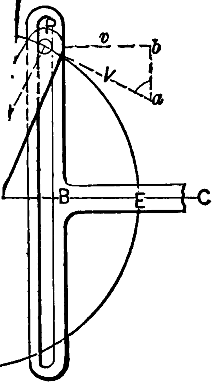


Fig. 106

$= \sin PAB,$

$PAB.$

when P is at D or E, we
or 270°, $v = V$.
tween 0 and V, and going
e in each revolution of AP,



lot is $2AP +$ diameter of pin.
is $2AP$.
used in some varieties of pumps
ift with the piston rod.

ed Sliding Bar. — In Fig. 110
a cam which, by its uniform rota-
imilar to that of Fig. 109, but with
st. The cam consists of a triangular
triangle being three equal arcs, each
point of intersection of the other two.
iformly about one of its corners

The follower is the slotted bar *BB*, and the cam acts upon the two straight edges of the slot, the distance between which is equal to the radius of curved edges of the cam.

Consequently the slot will be in contact with an angle and a side of the cam in every position, and the motion produced is as follows: Let the circle described by the outer edge of the cam be divided into six equal parts, as in the figure. Tracing the motion as the angle *m* of the cam goes round the circle in the direction of the numbers, it appears that no motion will be given to the bar while *m* is moving from 1 to 2. While *m* travels from 2 to 3, the face *Am* drives the upper side of the slot with an increasing radius; and hence the bar begins to move, and its velocity gradually increases. While *m* travels from 3 to 4 the action is similar to that of Fig. 109, and the motion of the bar will gradually be decreased until *m* reaches 4, when the bar will come to rest.

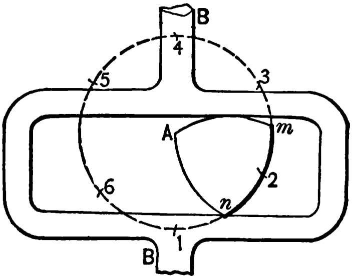


Fig. 110

As *m* moves from 4 to 5 the bar remains at rest; from from 5 to 6 the bar begins to move with an increasing velocity; from 6 to 1 the bar moves with a decreasing velocity, coming to rest as *m* reaches 1.

168. In case the direction of motion of the follower intersects the axis of motion of the cam, the latter may be made in the shape of a screw thread on a cone; when the follower's direction neither intersects nor is parallel to the cam axis we may employ a screw thread on a hyperboloid of revolution.

In fact, almost any kind of motion may be obtained by means of a suitably shaped cam; but the general principles employed in the various cases above treated of apply equally well to any other special cases.

169. Oldham's Coupling.—In Fig. 111 is shown a method of communicating equal rotation by sliding contact between two axes whose directions are parallel. *Aa* and *Bb* are the axes, each of which is furnished with a forked end, terminated by sockets bored in a direction to intersect the respective axes at right angles. The whole is so adjusted that all four sockets lie in one plane perpendicular to both axes. A cross with straight cylindrical arms is fitted into the sockets in the manner shown in the figure, and its arms are of a diameter that allows them to slide freely in their respective sockets. If one of the axes be made to revolve, it will drive the other with the same angular velocity.

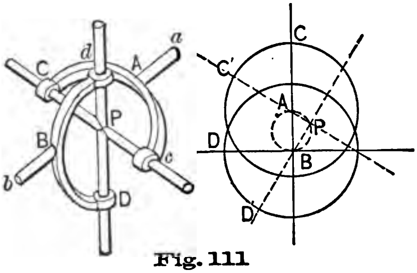


Fig. 111

For let the sketch at the right be a section through the cross perpendicular to the two axes *Aa* and *Bb*, and let the large circles be those described by their respective sockets. Then, if *C* be a socket of *Aa*, the arm of the cross which passes through it must meet the centre *A*; and in like manner, if *D* be a socket of *Bb*, the arm *DB* must pass through the centre *B*. Also, if *C* move to *C′*, the new position (dotted lines) of the cross will be found by drawing *C′A* through *A*, and *BD′* perpendicular to it through *B*. And it is evident that the angle *C′AC* = angle *D′BD*, hence the angular velocity is the same in both axes. In every position of the cross we will have the triangle *APB*, in which the side *AB* is constant, and the angle *APB* opposite to it is always a right angle. Hence the locus of *P* must be the circle whose diameter is *AB*; i.e., the centre of the cross will travel around the small dotted circle whose diameter is the distance between the axes. Also every arm will slide through its socket and back again during each revolution through a space equal to twice the distance between the axes.

In practice this arrangement is usually made in the shape of two discs, with a bar sliding in a diametral slit in each; the two bars being rigidly connected in the form of a cross.

170. An Escapement is a combination in which a toothed wheel acts upon two distinct pieces or pallets attached to a reciprocating frame, so that, when one tooth ceases to act on the first pallet, a different tooth shall begin to act on the second pallet. A simple example is shown in Fig. 112. The wheel *A* revolves continually in the direction of the

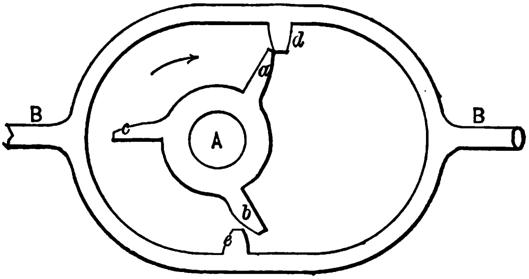


Fig. 112

arrow. The frame has two pallets, *d* and *e*, and can only move in the direction of its length. In the position shown, the tooth *a* is just *escaping* from the tooth *d*, and *b* is just ready to come in contact with *e*, by which the frame will be driven to the left. The shapes of the teeth may be designed as usual for a wheel and rack, and the point of quitting contact is found by the intersection of the addendum line of the wheel teeth with the describing circle of the pallets. The number of teeth on the wheel must evidently be *odd*.

But the frame may be used as the driver, instead of the wheel, by moving it alternately in each direction. This will cause the wheel to revolve in the opposite direction to that in which it would itself produce the reciprocation of the frame. But, when the frame is the driver, there is always a short interval at the beginning of each stroke, during which no motion will be given to the wheel.

171. Crown-wheel Escapement.—The crown-wheel escapement is used for causing the *vibration* of one axis by means of the uniform *rotation* of another. The latter carries a wheel consisting of a circular band, with large teeth, like those of a saw, on one edge. The vibrating axis, or *verge*, as it is often called, is located immediately above the crown wheel, and in a plane at right angles to the wheel axis, the latter being vertical. The verge carries two pallets, projecting from it in directions at right angles, and a sufficient distance apart so that they may engage alternately with teeth on opposite sides of the wheel. By this alternate action a reciprocating motion is set up in the verge. The rapidity of this vibration depends largely on the inertia of the verge, which may be adjusted by attaching a suitably weighted arm to the latter.

This escapement, though but rarely used at the present day, is of interest as being the first contrivance used in a clock for measuring time.

172. Anchor Escapement.—In Figs. 113 and 114 are shown two forms of this escapement. In Fig. 113 the wheel has long, slender teeth, and turns in the direction of the arrow. The vibrating axis *B* carries a two-armed piece having pallets *C* and *D* at its extremities, and resembling somewhat the form of an anchor, whence the name of the combination. When the tooth *g* presses against the pallet *C*, the normal at the point of contact passes on the same side of the centres *A* and *B*; hence (Art. 30) the tooth will tend

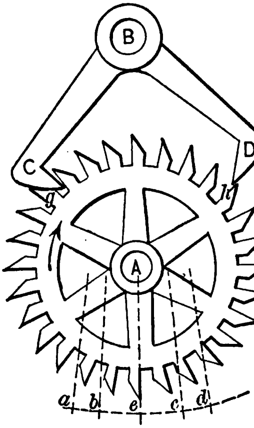


Fig. 113

to turn the pallet in the same direction as the wheel. *BC* will therefore turn upwards, and allow the tooth to escape from the pallet. At this instant the tooth *k* will begin to act on the pallet *D*; and, as the normal here passes between the centres *A* and *B*, *BD* will move in opposite direction to the wheel, and hence the tooth *k* will escape.

The teeth in an anchor escapement are often replaced by pins, in which case the form of the anchor may be so altered that the action shall take place entirely on one side of the line of centres, as shown in Fig. 114. The rapidity of vibration is controlled by the inertia of a weight or pendulum attached to the verge. This very inertia, however, preventing the verge from being *suddenly* stopped and reversed in direction, causes a recoil action to be set up in the wheel, which materially diminishes the utility of this escapement; for it is evident that, as the verge cannot be stopped suddenly, the wheel must of necessity give way and recoil at the first instant of each engagement between a tooth and its corresponding pallet. The greater the inertia due to the load attached to the verge, the more slowly will the escapement work, and the greater will be the amount of recoil.

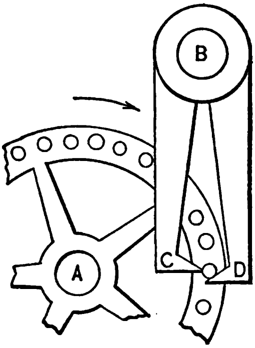


Fig. 114

173. Method of connecting Anchor and Pendulum.—There is one uniform method of connecting the anchor and the pendulum, which can be seen in almost any clock. The pendulum, consisting often of a compound metal rod with a heavy bob, is swung by a piece of flat steel spring, and vibrates in a vertical plane very near to that in which the anchor oscillates. To the centre of the anchor is attached a light vertical rod, having the end bent into a horizontal position, and terminating in a fork which embraces the pendulum rod. It follows that the anchor and the pendulum swing together, though each has a separate point of suspension.

174. Action of Escapement on Pendulum. — In Fig. 113, let the escape wheel tend to move in the direction of the arrow, so as to press its teeth slightly against the pallets of the anchor; the pendulum being hung from its point of suspension by a thin strip of steel, and vibrating with the anchor in the manner already stated. Let the arc *abecd* be taken to represent the arc of swing of the centre of the bob of the pendulum. As the pendulum moves from *d* to *b*, the point *g* of the escape wheel rests upon the oblique lower surface of the pallet *C*, and presses the pendulum onward until the latter reaches *b*, when the point of the tooth escapes at the end of the pallet. For an instant the escape wheel is free; but a tooth is caught at once upon the opposite side by the oblique upper surface of *D*, and the escape wheel then presses against the pendulum, and tends to stop it, until finally the pendulum comes to rest at the point *a*, and commences the return swing. During the latter the pendulum is similarly at first urged on, and then held back by the action of the escape wheel.

This alternate action *with* and *against* the pendulum prevents the pendulum from being, as it should be, the exclusive regulator of the speed of revolution of the escape wheel; for its own speed, instead of depending solely on its length, will also depend on the force urging the escape wheel round. Hence any variation in the maintaining force will disturb the rate of the clock.

175. Dead-beat Escapement. — This objectionable feature is obviated in Graham's *dead-beat* escapement, Fig. 115. It is, however, most worthy of note that the change in construction which abolishes the defects due to the recoil, and gives the astronomer an almost perfect clock, separates the combination entirely from its original conception; viz., that of an apparatus for converting circular into reciprocating motion. The improvement consists in making the lower surface of the pallet *C* and the upper surface of the pallet *D* arcs of circles, whose centre is at *B*. The oblique surfaces *qm*, *np*, complete the pallets. As long as the tooth is resting on the upper surface of *D*, the pendulum is free to move, and the escape wheel is locked; hence in the portion *ba* of the swing, and back again through *ab*, there is no action against the pendulum except the very minute friction which takes place between the tooth of the escape wheel and the surface of the pallet. Through the space *bc* the point of the escape wheel tooth is pressing against the oblique edge *np*, and is urging the pendulum forward.

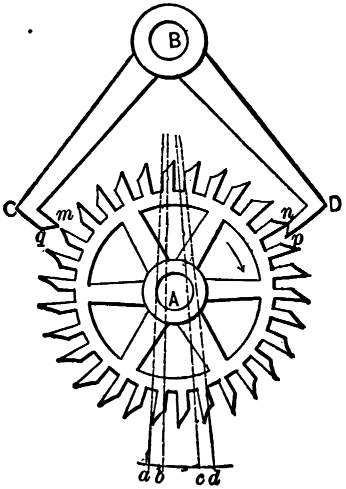


Fig. 115

Then at *c* this tooth escapes, and the tooth upon the opposite side falls upon the lower surface of *C*, and the escape wheel is locked; from *c* to *d*, and back again from *d* to *c*, there is the same friction which acted through *ba* and *ab*. From *c* to *b* the point of a tooth presses upon *gm*, and urges the pendulum onward; at *b* this tooth escapes, another one comes into contact, and so on. It follows that there is no recoil, and the only action *against* the pendulum is the very minute friction between the teeth and the pallets. The term "dead-beat" has been applied because the seconds hand, which is fitted to the escape wheel, stops so completely when the tooth falls on the circular portion of the pallet. There is none of that recoil or subsequent trembling which occurs in the other escapements.

CHAPTER X.

COMMUNICATION OF MOTION BY LINKWORK.

VELOCITY RATIO AND DIRECTIONAL RELATION CONSTANT OR VARYING.

Classification.—Discussion of Various Classes.—Quick Return Motion.—Hooke's Coupling.—Intermittent Linkwork.—Ratchet Wheels.

176. As has been shown by the general definition (Art. 22), linkwork derives its name from the rigid connecting piece or link. This connecting piece is known by various names under different circumstances, such as *connecting-rod*, *coupling rod*, *side rod*, *eccentric rod*, etc. The arms are known as *cranks* when they perform complete revolutions; and as *beams*, *crank arms*, *rocker arms*, or *levers*, when they oscillate.

177. Classification of Linkwork.—Linkwork is used,

I. To transform circular motion into rectilinear reciprocation, or the reverse.

II. To transform continuous rotation into rotative reciprocation, or the reverse.

III. To transmit continuous rotation.

Examples of the first class are seen in slotting and shaping machines, power pumps, and in the usual forms of the steam engine; of the second class, in steam engine valve motions, where a rocker shaft is employed; and of the third class, in locomotive side rods.

CLASS I.

Transformation of Circular Motion into Rectilinear Reciprocation, and the Reverse.

178. In Fig. 116, let AP be a crank revolving about the fixed centre A, and connected by a link PQ to a point Q, travelling in a straight line KL whose direction passes through the centre A. Let $AP = R$, and $PQ = l$. The length of the path of Q is evidently equal to $2R$. When P is at C or D, the points A, P, Q, will be in one straight line. The points C and D are called *dead points;* since when P is at either of them, the revolution of AP will cause no motion whatever to be transmitted to the point Q, for that instant. When PQ overlaps AP, as when P is at D, we shall term the point D the *inward* dead point; and when Q lies at the other extremity of its stroke, so that P is at C, we shall term the point C the *outward* dead point. In Fig. 116, let fall from P the line PE perpendicular to AQ. Then the distance of Q from A is at any instant, $AQ = QE + AE = \sqrt{l^2 - R^2 \sin^2 \theta} + R \cos \theta$, the last term of which will be essentially negative when θ lies between 90° and 270°.

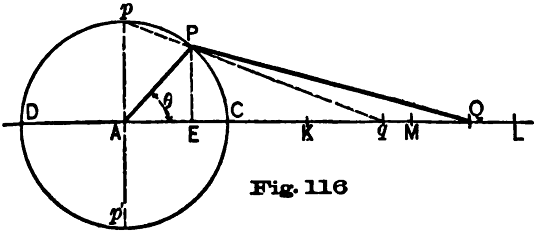


Fig. 116

If PQ were of infinite length, the motion of Q would be equal to that of the point E; but as PQ is of finite length (usually from four to eight times AP), Q is drawn toward A through the distance $PQ - EQ = l - \sqrt{l^2 - R^2 \sin^2 \theta}$. So that when AP has moved to its mid-position Ap or Ap' (or, as it is frequently expressed, AP is on the half-centre), Q will have passed its mid-position M by the distance $qM = AM - Aq = l - \sqrt{l^2 - R^2}$. Also when Q is at M, P will be at some point S or S' (Fig. 117), intermediate between C and p or p'. These points may be readily determined, for in this case $AQ = l$; hence AQS and AQS' are equal isosceles triangles, and $\cos\theta = \frac{AD}{AQ} = \frac{R}{2l}$. The velocity ratio of P and Q varies for each instant, but may be determined at any time by means of the instantaneous centre (Art. 25) or by resolving the velocities, as in Fig. 118.

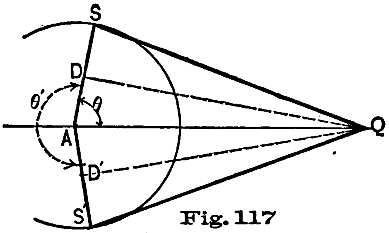


Fig. 117

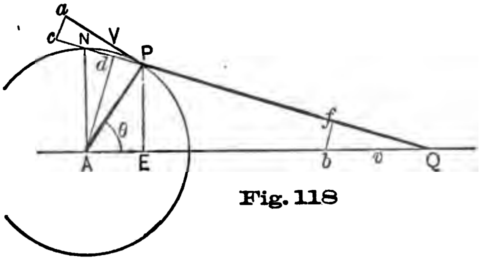


Fig. 118

Let V be the linear velocity of P, and v that of Q. Resolve these along and perpendicular to the link PQ; then, as shown in Art. 24, the components along the link must be equal; that is,

$$V \cos aPc = v \cos bQf. \qquad (1)$$

Draw AN perpendicular to AQ, and intersecting the link (produced) at N; draw Ad perpendicular to PQ. Then angle $aPc =$ angle PAd; and angle $bQf =$ angle NAd. Hence

$$\cos aPc = \cos PAd = \frac{Ad}{AP}, \qquad (2)$$

and

$$\cos bQf = \cos NAd = \frac{Ad}{AN}. \qquad (3)$$

Substituting these values in equation (1), we get

$$V \times \frac{Ad}{AP} = v \times \frac{Ad}{AN}. \qquad (4)$$

Hence,

$$\frac{v}{V} = \frac{AN}{AP} = \frac{AN}{R}, \qquad (5)$$

a variable quantity. It is evident from this expression that when $AN = R$, the velocities of P and Q are the same. This will occur when AP is perpendicular to AQ, as at Ap, Ap' (Fig. 116), in which case AP coincides with AN; and it will also occur when AP occupies such a position that the triangle APN is isosceles. To determine the angle θ which will give this position of AP, we have, from similar triangles, $AN : PE :: AQ : EQ$.

$$\therefore AN = R = PE \times \frac{AQ}{EQ} = R \sin\theta\left(\frac{\sqrt{l^2 - R^2\sin^2\theta} + R\cos\theta}{\sqrt{l^2 - R^2\sin^2\theta}}\right).$$

From this equation we deduce

$$\sin\theta = \frac{l}{4R^2}(\sqrt{8R^2 + l^2} - l).$$

179. The distance through which the point Q is drawn toward A by reason of the finite length of the link (Art. 178) increases rapidly as the link becomes shorter. If we make the link of the *same length* as the crank arm, as in Fig. 119, the point K (Fig. 116) coincides with A, and the path

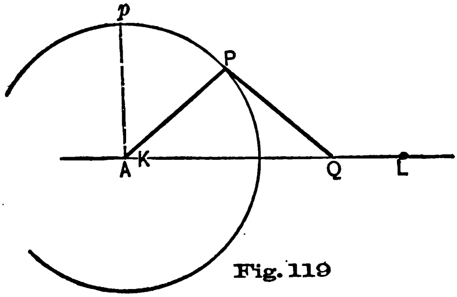


Fig. 119

of Q is $AL = 2R$, as before. But Q is drawn toward AP so rapidly on account of the angularity of PQ, that when

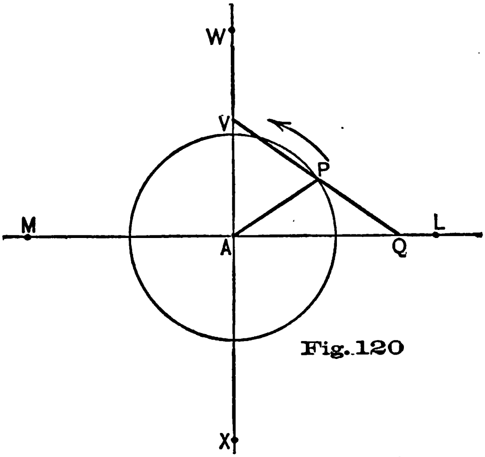


Fig. 120

AP is perpendicular to AL, Q coincides with A and has completed its stroke. If we produce QP to V, making

PV = *PQ* = *AP* (Fig. 120), it is evident that as *AP* revolves as indicated by the arrow, *Q* will move from *L* to *A*, and *V* will move from *A* to *W*. If now we continue the motion of *AP*, *Q* will be driven past *A* to *M*, and *V* will return to *A*. Thus, the revolution of *AP* will cause *Q* to move over the path *LM*, and *V* over the path *WX*. By this means, the arm *AP* can be made to move the two ends of a link of twice its length through paths at right angles, and each equal in length to 4*AP*.

180. We have thus far considered the end of the link, *Q*, to travel in a path of which the direction passes through the axis *A*, but this path may be a straight line *not* passing through *A*, as in Fig. 121. In this figure, let *AP* be a crank revolving about the centre *A* and connected by the link *PQ* to the point *Q* travelling in the straight line *KL*. An arc of a circle struck about centre *A*, with radius $l - R$, will cut the line *KL* at *K*, the end of the stroke; and the *inward* dead point *D* will lie in the straight line *KAD*. Similarly, the other end of the stroke, *L*, and the *outward* dead point *C*, may be found by striking an arc about centre *A*, with radius $AL = l + R$.

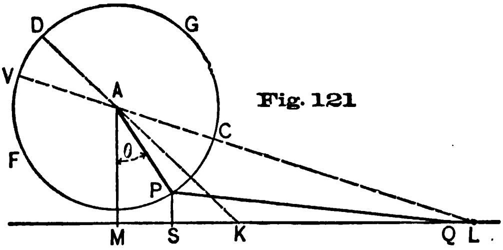


Fig. 121

The position of the point *Q* corresponding to any given position of *AP* may be thus determined:—

Let the perpendicular distance between A and the line KL be $AM = e$, and let the angle $PAM = \theta$. Then

$$MQ = MS + SQ, \quad MS = R \sin \theta;$$

$$SQ = \sqrt{l^2 - Ps^2} = \sqrt{l^2 - (e \pm R \cos \theta)^2}.$$

Hence we have

$$MQ = R \sin \theta + \sqrt{l^2 - (e \pm R \cos \theta)^2}.$$

Also

$$ML = \sqrt{(l + R)^2 - e^2}.$$

From these expressions, the distances QL and QK can be determined.

181. By comparing Figs. 116 and 121, it will be seen, that, in the former, the outward and inward dead points, C and D, lie in one straight line; while in the latter, they depart from a straight line by the angle $DAV = KAL$. This angle increases as AM is increased, and as the ratio $\frac{R}{l}$ is increased. The practical result is, that, supposing AP to revolve uniformly in the direction of the arrow, the point Q will move over its path from L to K in less time than from K to L, the times being proportional to the arcs DGC and DFC.

182. Eccentric. — A particular form of this class of link motions is the eccentric, shown in Fig. 122. A circular disc called the *eccentric*, or the *eccentric sheave*, has its centre at the point P, and is made sufficiently large to embrace the shaft at A, to which it is fastened. The eccentric is enclosed by a *strap* or *band*, FG, in which it revolves. This strap is rigidly connected to the *rod* or *bar* HN, by which motion is transmitted to the point Q. It will be seen, that, as the eccentric turns about A and slides within the strap, it will communicate exactly the same motion to the point Q as would be given by a crank arm AP and link PQ. In fact, it is used as a substitute for small cranks on account of the practical difficulties in the formation of the latter. The travel of the point Q will, as in Fig. 116, be equal to $2AP$.

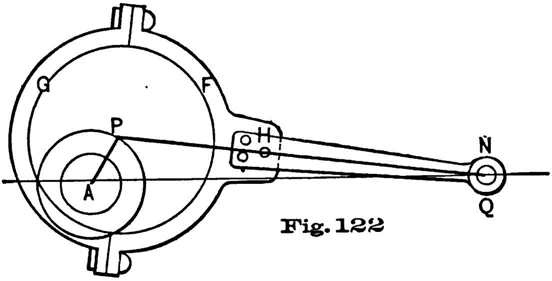


Fig. 122

The term, *throw* of an eccentric, is given, by various authorities, either to the arm AP, or to twice that distance; and hence the meaning of the term is often ambiguous.

CLASS II.

Transformation of Continuous Rotation into Rotative Reciprocation, and the Reverse.

183. In Fig. 123, let AP and BQ be arms turning about the fixed centres A and B respectively, and connected by the link PQ. If AP be rotated about A, it will compel BQ to *oscillate* between the positions Bc and Be, or Bc' and Be', according as the arm BQ has been previously placed above or below the centre B.

Let $AP = R$, $BQ = r$.

$PQ = l$, $AB = d$.

To find the dead points: About A as a centre describe circular arcs with radii $l + R$ and $l - R$. They will cut the circle about centre B, radius r, in the points c, c', and e, e', respectively. These give the outward and inward dead points for R, and hence the limits of the oscillation of r. Drawing the pieces in these extreme positions, it will be seen that we obtain a series of triangles, of which the base is always the line of centres AB $(= d)$, and of which the other two sides are r, and $l + R$ or $l - R$. We will term these *dead-point triangles*. As long as we can construct such triangles with a sensible altitude, it is clear that there can be no dead points for r, and hence the rotation of R will cause r to oscillate. But if with any assumed values of R, l, and r, the triangle will reduce to a straight line, r will have dead points, and we can no longer control the *direction* of its motion by the single combination shown. Thus, in order that the rotation of an arm R may produce oscillation of an arm r, we must have r greater than R, and also

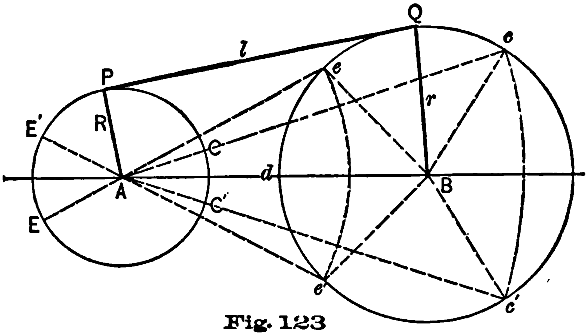


Fig. 123

$$d + r > l + R.$$

$$d - r < l - R.$$

These conditions are fulfilled by the proportions employed in Fig. 123.

184. Figs. 124, 125, and 126 are inserted to show the effects of shortening the distance between centres, retaining the same lengths of R, l, and r, as in Fig. 123.

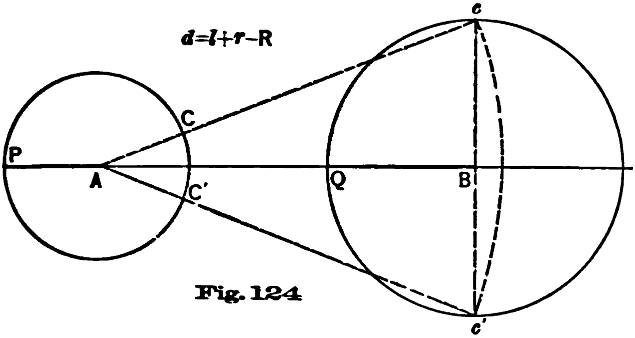


Fig. 124

In Fig. 124, $d = l + r - R$, and we have an outward dead point for r, simultaneously with an inward dead point for R. In Fig. 125, $d = l + R - r$, and we have an inward dead point for r, simultaneously with an outward dead point for R. These two arrangements are therefore faulty; since when r is at a dead point, we cannot control the direction of its motion, as it is free to move either up or down

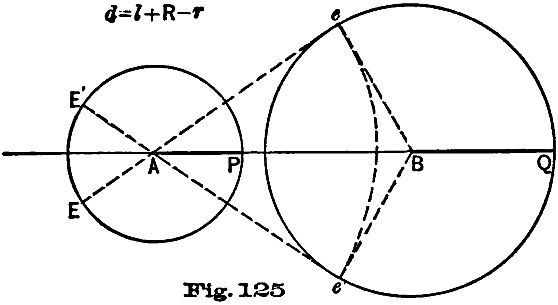


Fig. 125

from that point. In Fig. 126, *d* is made still shorter, and it will be seen that *r* has dead points, *G* and *G′*, when *R* is at *Ag* or *Ag′* respectively. Hence *R* cannot perform complete revolutions and drive *r*.

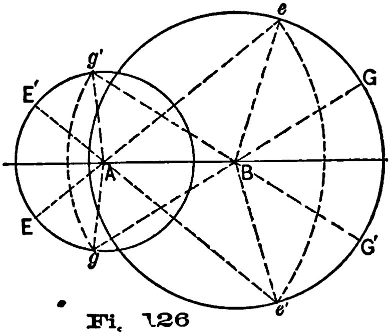


Fig. 126

As *R* moves in either direction over the arc *gEE′g′*, *r* may move in either direction over the path *G′e′G′G* or *GeGG′*.

185. Problem.—Given (Fig. 127) the distance *AB* between two fixed centres *A* and *B*, to find the lengths of

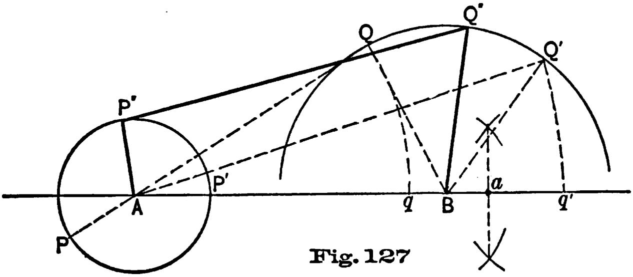


Fig. 127

arms and links which will cause the following arm *BQ* to oscillate between the positions *BQ* and *BQ′* when the driving arm *AP* makes continuous rotations. Assume any convenient length for the arm *BQ*, and strike an arc with that length as a radius about B, thus fixing the points Q and Q'. With A as a centre, draw the circular arcs Qq and $Q'q'$. Then we will have $AQ = Aq = l - R$; and $AQ' = Aq' = l + R$; $\therefore\ qq' = 2R$. Bisect qq' at a, then aq or aq' is the length of the arm AP to fulfil the conditions, and Aa is the length of the link $PQ = l$. If these results do not give convenient lengths for R and l, a longer or a shorter length should be taken for BQ.

186. The velocity ratio in this class of linkwork is to be determined, as in Arts. 24 and 25, by the ratio of the perpendiculars let fall from the fixed centre upon the line of the link. When there are dead points, one or both of the perpendiculars disappear at the instant of passing these points; and this is just as it should be, for *no* motion is transmitted *at that instant.*

When the shorter and rotating arm is the driver, its dead points occur, as shown in the figure, at the ends of the oscillations of the following arm, and this arrangement will work satisfactorily when the conditions of Art. 184 are complied with. In case the oscillating arm drive, however, the dead points of the follower must be overcome by the momentum of the rotating pieces, increased if necessary by the addition of a fly-wheel.

CLASS III.

Transmission of Continuous Rotation.

187. Drag Link.—In Figs. 128 and 129, let AP, BQ, be two arms turning about fixed centres A and B, and connected by the link PQ, as before. In order that the continuous rotation of one may produce a continuous rotation of the other, it is necessary that there shall be *no dead points.* If the link PQ $(= l)$ is made equal to Cc, it is evident that we will have an outward dead point of R at C with an inward dead point of r at c. Or, if $l = Ce$, we will have simultaneous inward dead points at C and e. Therefore, in order that there may be no dead points, we must make $l > Cc$ and $< Ce$;

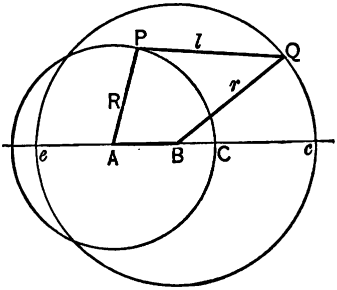


Fig. 128

that is to say, $l > r - R + d$, and $l < r + R - d$; and each of the arms R and r must be greater than d.

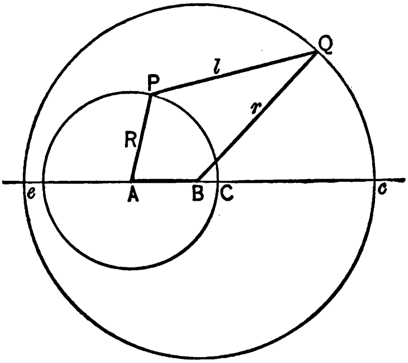


Fig. 129

In this arrangement, the arms may be equal or not; but in either case the velocity ratio, being proportional to the perpendiculars from the centres on the link, is constantly varying. The arrangement is termed a *drag link*. As frequently constructed, the arms are of the same length, and the centres A and B coincide.

188. Continuous rotation may be transmitted by making the arms R and r of Fig. 123 equal, and also $l = AB$. This will give us the arrangement shown in Fig. 130. The arms have simultaneous dead points when lying in the direction of the line of centres. Hence if R performs complete revolutions, r may continue to rotate in the same direction, with a *constant* velocity ratio, or it may move from the dead point in the opposite direction, with a varying velocity ratio; so that for a given position, AP, of R, r may occupy the position BQ or Bq. To insure continuous rotation, then, by this arrangement, it is necessary to provide some means of compelling r to continue its motion past the dead points. This may be accomplished by one of the following arrangements.

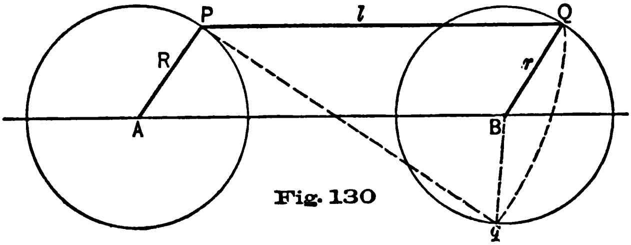


Fig. 130

189. We may connect the axes by other and similar systems, as shown in Figs. 131 and 132.

When two systems of arms and links are employed, as in Fig. 131, they are generally placed with the arms at right angles.

When three are used, as in Fig. 132, the arms are placed at angles of 120°.

190. Another method consists in the use of a third rotating arm, connected to the same link, and so placed that the driving arm may lie between the following and the auxiliary

Fig. 131

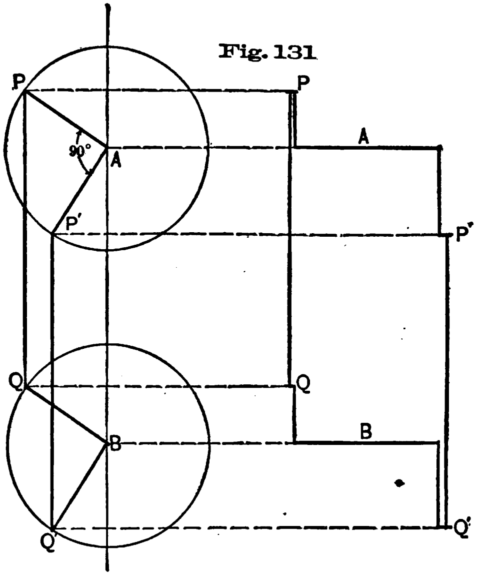
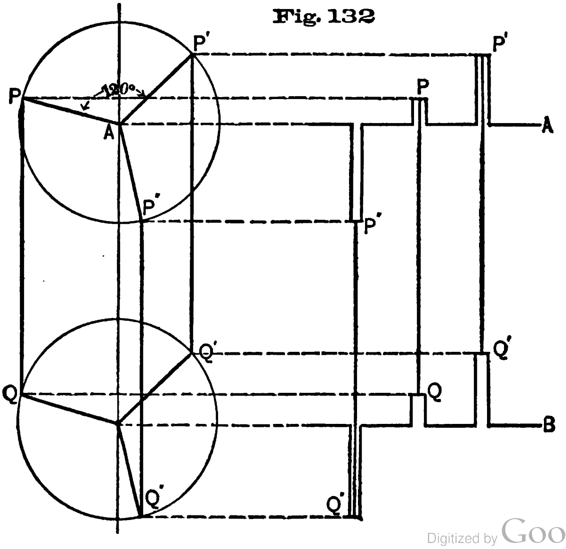


Fig. 132



arms. This arm must be equal in length to the other two, and must lie parallel to them in all positions. All three arms may be located on the same line of centres, as in Fig. 133; or the driving arm may be to one side of the line of centres of the other two, as in Fig. 134. In the latter case *VT* must be part of or be rigidly connected with *PQ*, so that the points *P*, *V*, *Q* will be the vertices of a rigid triangle.

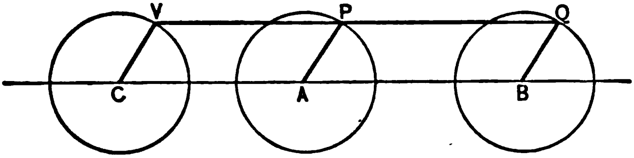


Fig. 133

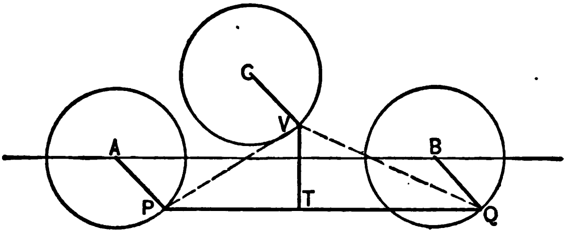


Fig. 134

191. Boehm's Coupling. — Another method, known as Boehm's link coupling, is shown in Fig. 135. Two discs placed in parallel planes are fixed to parallel shafts, and connected by two or more links which make an angle with the planes of the discs. The distance between the planes of the discs must be sufficient to enable the links to pass each other, as shown. The velocity ratio in these three arrangements is constant.

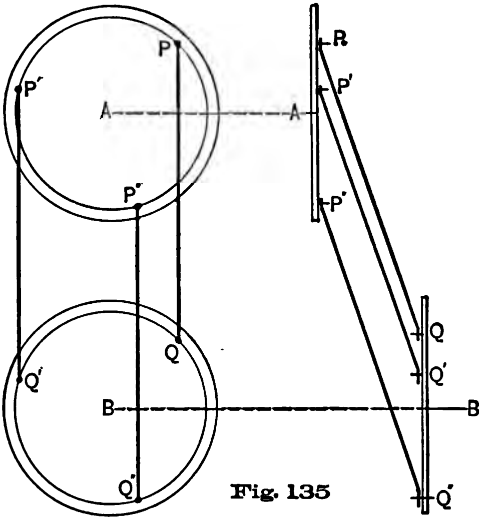


Fig. 135

VARIOUS APPLICATIONS OF LINKWORK.

192. Bell-Cranks.—A form of linkwork known as bell-cranks is largely used to change the direction of motion. In Fig. 136 let *ab* be the direction of a reciprocating motion which it is desired to change to the direction *cd*. In the angle *bTd* formed by these directions assume any convenient point *A*; and from *A* draw perpendiculars *AB* and *AC* on *ab* and *cd* respectively. If we construct a rigid piece *BAC*, centred at *A*, we can by means of it produce the change in direction desired. This piece is termed a bell-crank, and such pieces are largely used in bell-hanging, in the mechanism of organs, etc. As the angular motion of the arms is small, their lengths are sensibly equal to the perpendiculars from *A* upon the lines of action, and hence the velocity ratio is sensibly constant.

It is clear that we may place the centre *A* in any one of the four angles about *T* made by the lines of action. If placed in the angle *bTd* or *aTc*, the direction of motion will be as indicated by the arrows; but if we place it in the angles *aTd* or *bTc*, the direction of motion along *ab* being still as indicated by the arrow, that along *cd* will be reversed.

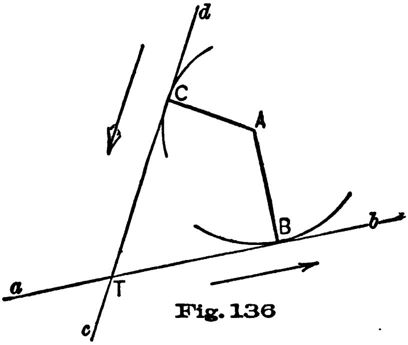


Fig. 136

193. In order that the deviation of the points *B* and *C* from the lines of action shall be a minimum, and take place on both sides of the line of action instead of wholly on one side, the length of the arm *AB* should be equal to the perpendicular distance *AB* *plus* one-half the versed sine of the angle through which the arm swings on each side of its mid-position. This is true in all similar cases, and is illustrated by the following example: In a beam engine having a piston stroke of six feet, let the distance between the centres *A* of the beam gudgeons, and the centre line *BP* of the cylinder be eight feet. Required, the best length for the beam arm. In Fig. 137 let *AB* = distance between centres = 8′. Now *B* is to bisect the length *CD*, which is the sum of the deviations of the point *E* on both sides of *BP*; hence *BC* = *BD*. *EC* must equal the half stroke, or 3′.

In the right triangle ECA, we have

$$\overline{EA}^2 = \overline{EC}^2 + \overline{CA}^2 = \overline{AD}^2$$

or $(8 + BC)^2 = 9 + (8 - BC)^2$. Solving this equation for BC, we find $BC = \frac{9}{32}' = \frac{27}{8}'' = 3\frac{3}{8}''$.

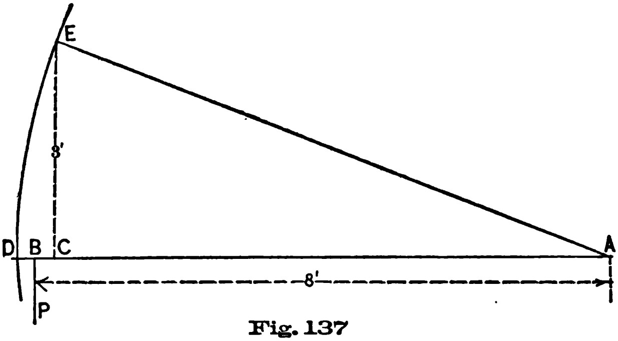


Fig. 137

Therefore the length of the beam arm AE should be $8'\ 3\frac{3}{8}''$, in order that the connection E may deviate the smallest amount from the line of centres BP.

194. To Multiply Oscillations by Linkwork. — In Fig. 138, let AP be an arm rotating about A, and connected by a link PC to an arm BC oscillating about B. For one revolution of AP, BC will move from the position BC to Bc and back again to BC. Now connect C by a link CE to an arm DE oscillating about D. Let AP turn from P to p, and move BC from C to c, then DE will be moved from E to E' and back again to E. Therefore during a *complete* revolution of AP, DE will perform *two* complete cycles of motion; going from E to E' and back again to E during each *half* revolution of AP. If we connect DE to another oscillating arm KH in a similar manner, KH will perform two complete cycles for one of DE, and hence *four* complete cycles for one revolution of AP.

The length of the arc of oscillation of each arm depends on the length of the preceding arm, and on the versed sine of the angle through which the latter swings on each side of its mid-position.

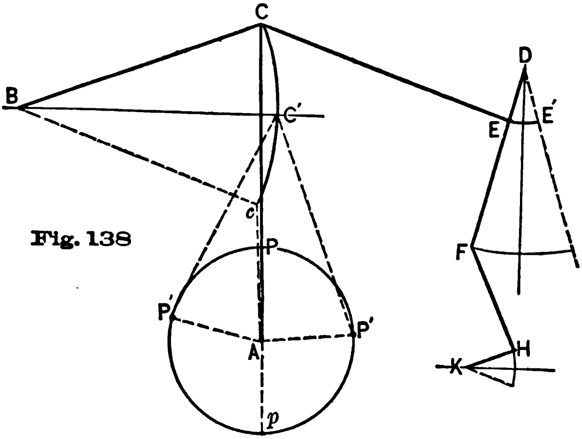


Fig. 138

195. To Produce a Rapidly Varying Velocity from Uniform Motion. — In Fig. 139 let *AP* be an arm oscillating with uniform velocity about *A*, and connected by a link *PQ* to an arm oscillating about a centre *B*, placed so

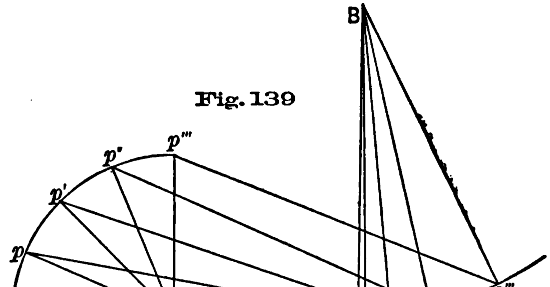


Fig. 139

that when *P* is at a dead point, *BQ* may be perpendicular to *PQ*. Starting from the dead point *P*, the uniform motion of *AP* produces very little motion of *BQ* at first, but as *P* moves over the equal arcs *Pp*, *pp'*, etc., *Q* will move through arcs rapidly increasing in length, such as *Qq*, *qq'*, etc. That is, the uniform velocity of *P* produces a rapidly accelerated velocity of *Q*. When *P* moves in the other direction, from *p'''* toward *P*, the velocity of *Q* will be rapidly retarded.

196. Slow Advance and Quick Return by Linkwork. — In Fig. 140, let *AP* be a rotating arm, from whose uniform motion we wish to derive the motion of a second arm, such that its period of advance shall be equal, say, to twice its period of return. On a circle about *A*, lay off the arc *PCp* = one-third of the circumference; i.e., angle *PAp* = 120 degrees; then *AP* and *Ap* must be the positions of the driving arm when the following arm is at the ends of its stroke, and the driver is at its dead points.

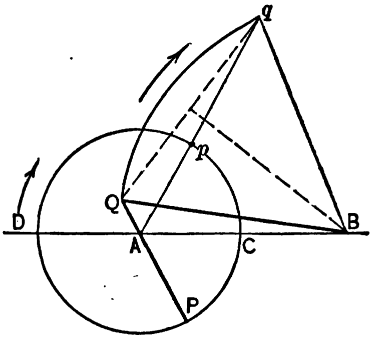


Fig. 140

On *PA* and *Ap* produced, lay off *PQ* = *pq* = proposed length of link. Then *Qq* must be the chord of the arc of oscillation of *BQ*, and the centre *B* must lie in the perpendicular bisecting this chord. *BQ* may be of any length to give the angle of oscillation QBq desired, consistent with the deductions of Art. 184.

197. Hooke's Coupling or Universal Joint is a contrivance, belonging to the general class of linkwork, for connecting shafts whose axes intersect.

In Fig. 141, A and B are the two shafts having semicircular jaws at their ends. The connecting and rigid cross

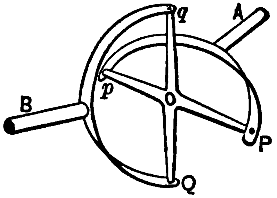


Fig. 141

$QPpq$ is formed with its four arms at right angles and in the same plane. The centre of the cross is at O, the intersection of the axes. All four arms are of the same length, and turn in bearings at P, p, Q, and q. Let the shaft A be the driver, then the ends of the arms, P and p, move in a circle whose plane is perpendicular to the axis of A; and the ends, Q and q, move in a circle whose plane is perpendicular to the axis of B. We will term these arms the driving and following arms respectively. The planes of the circles evidently intersect in a line through O, perpendicular to the plane of the axes, and the angle between the planes is equal to the angle between the axes $= \beta$.

198. Let a plane through O, and perpendicular to the driving axis, be taken as the plane of projection. Then, Fig. 142, the circle described with radius $OP = Op$ about O as a centre, will represent the path of the points P and p. Let the plane of the axes A and B be perpendicular to the paper, intersecting it in OE. Then COD, perpendicular to OE, will be the intersection of the planes of rotation of the driving and following arms. Draw the radius OF, making the angle $EOF = \beta =$ the acute angle between the axes; and from F draw a line parallel to COD, and intersecting OE in K. Then

$$\frac{OK}{OF} = \frac{OK}{OE} = \cos\beta,$$

and an ellipse constructed with CD as its major and OK as its semi-minor axis will represent the projection, on a plane perpendicular to the driving axis, of the path of the following arms.

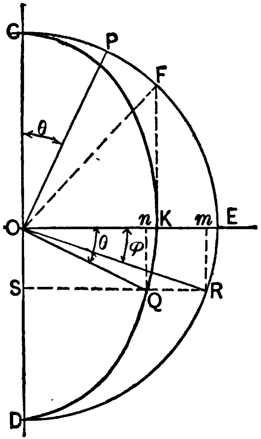


Fig. 142

Suppose a driving arm to move from the position OC to OP through the angle $COP = \theta$. Then a line OQ drawn perpendicular to OP will be the projection of a following arm which has moved from OK, while OP moved through the angle $COP = EOQ = \theta$. OQ is perpendicular to OP, since the latter lies in the plane of projection; and hence the angle POQ is shown in its true size. The point Q has moved through the actual vertical distance Qn, although the actual path of Q is a circle of radius equal to OE. Therefore, if through Q we draw a line parallel to OE, and connect OR, then EOR will be the actual angle through which the following arm has moved, while the driving arm has moved through $COP = EOQ = \theta$. Let angle $EOR = \phi$.

$$\frac{\tan\phi}{\tan\theta} = \frac{mR}{Om} \div \frac{nQ}{On} = \frac{On}{Om} = \frac{SQ}{SR} = \frac{OK}{OE} = \cos\beta.$$

Hence

$$(1) \qquad \tan\phi = \tan\theta\cos\beta.$$

To obtain the velocity ratio, we must differentiate this expression, whence

$$(2) \qquad \frac{d\phi}{d\theta} = \frac{\cos^2\phi}{\cos^2\theta}\cos\beta = \cos\beta\frac{1+\tan^2\theta}{1+\tan^2\phi} = \frac{a'}{a}.$$

Eliminating ϕ and θ in turn from Equation (2) by means of Equation (1), we get

$$(3) \qquad \frac{a'}{a} = \frac{\cos\beta}{1-\sin^2\theta\sin^2\beta}.$$

$$(4) \qquad \frac{a'}{a} = \frac{1-\cos^2\phi\sin^2\beta}{\cos\beta}.$$

Starting with a driving arm at OC and a following arm at OK, we measure the angles θ and ϕ from these positions respectively.

199. The expressions (3) and (4) will have minimum values when $\sin\theta = 0$, and $\cos\phi = 1$; in that case $\frac{a'}{a} = \cos\beta$, and θ and ϕ both $= 0, \pi, 2\pi$, etc. That is, the minimum values of the velocity ratio occur when a driving arm is at OC or OD, and the following arm is at OK or KO produced.

Maximum values occur when $\sin\theta = 1$; $\cos\phi = 0$. In this case $\frac{a'}{a} = \frac{1}{\cos\beta}$, and θ and ϕ both $= \frac{\pi}{2}, \frac{3\pi}{2}$, etc. That is, the velocity ratio has its maximum value when the driving arm is at *OE* or *EO* produced, and the following arm is at *OD* or *OC*.

Hence we see that during each revolution there are two maximum and two minimum values of the velocity ratio, and that it varies between the values $\frac{1}{\cos\beta}$ and $\cos\beta$.

Between the maximum and minimum points there are four points where the ratio is unity.

The variation in the velocity ratio increases as the angle between the axes increases, and this fact must, in general, govern the employment of this joint. If the variation due to the angle between the shafts is not too great for the case under consideration, it may be employed; otherwise some other mode of connection must be used.

200. Double Hooke's Joint. — The variation in the velocity ratio may be entirely eliminated by the use of two Hooke's joints, arranged as in Fig. 143.

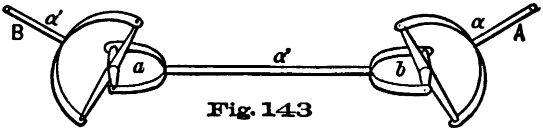


Fig. 143

If we construct the connecting piece with the forks at its ends *in the same plane*, and place it so that it makes the same angle (β) with both shafts *A* and *B*, the variation at one end of the connecting piece will be counterbalanced by the variation at its other end; and thus a uniform motion may be transmitted from *A* to *B*. For, in Fig. 143, let the plane of the axes be the plane of the paper, then considering

A as the driver, the velocity ratio between *A* and *ab* is at its maximum; i.e.,

$$\frac{a''}{a} = \frac{1}{\cos\beta}.$$

Also, considering *ab* as the driver, the velocity ratio between *ab* and *B* is at its minimum; i.e.,

$$\frac{a'}{a''} = \cos\beta.$$

Multiplying these two equations together, we get

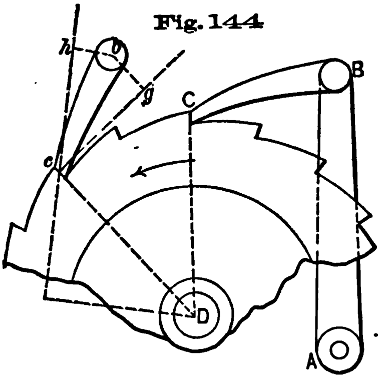
$$\frac{a'}{a} = 1.$$

It will be clear upon examination that the variations thus balance each other throughout the revolution, so that $\frac{a'}{a}$ is always unity; in other words, the velocities of the two principal axes are always equal. If the cross ends of *ab* are in planes perpendicular to each other, the variations, instead of neutralizing each other, will evidently act together, and make the total variation in velocity of the two principal axes greater than if one joint only were employed.

INTERMITTENT LINKWORK.

201. Click and Ratchet. — An example of intermittent linkwork is the *click* and *ratchet-wheel*, a simple form of which is shown in Fig. 144. An arm *AB* oscillating about *A* has jointed to it at *B* a *click* or *catch BC*. Turning about *D* is a *ratchet-wheel Cc*, having teeth generally of the shape shown. When the arm *AB* moves as indicated by the arrow, the end *C* of the click presses against the straight side of a tooth, and thus moves the wheel. Centred at the *fixed* point *b* is a *pawl* or *detent*, *bc*, which either rests on the teeth by its weight, or is pressed against them by a spring. This serves, as shown, to hold the wheel, and prevent its backward motion during the back stroke of *AB*, but offers little or no resistance during the forward motion of *AB*. The pawl and tooth act upon each other by sliding contact; and the direction of the line of action, or of the pressure between them, is a normal to the straight face of the tooth and end of the pawl.

Fig. 144



Let *cg* be this normal, and let fall upon it the perpendiculars *bg* and *Dc*. Then, if the wheel tend to turn backwards, that is, in the direction *cg*, the pawl will tend to turn about *b* in the same direction, *cg*, or towards the wheel. That is, the tendency is to force the pawl and the tooth into closer contact, which is as it should be. But if the shape of the face of tooth and pawl is such that the normal occupies some position beyond *b*, as *ch*, the tendency is to turn the pawl outward, or to cause it to slide off the tooth. Hence, with a straight pawl or click, as shown in this figure, the normal to the face of the tooth should pass between the centres of the wheel and of the pawl. And, by similar reasoning, in the cases in which we have a *hooked* pawl, as in Fig. 148, the normal should pass outside of or beyond the centre of motion of the pawl.

202. Reversible Click. — In feed motions, such as those of shapers and planers, it is frequently desirable to employ a click and ratchet-wheel which will drive in either direction. An arrangement similar to that shown in Fig. 145 is then used. In the position shown, the wheel is being driven in the direction of the arrow. The teeth, as well as the click, are made alike on both sides; so that when the click is thrown over on the other side of the arm *AB*, as shown in dotted lines, the wheel will be driven in the contrary direction.

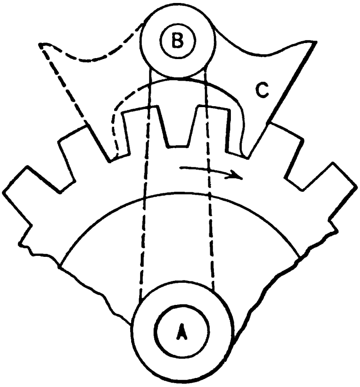
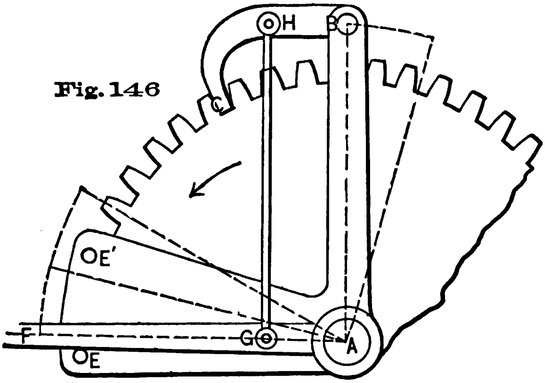


Fig. 145

203. It is usually more convenient to have the driving arm and the wheel concentric, as in Fig. 145, rather than as in Fig. 144. It is clear, that, in this case, to have an effective arrangement, the driving arm should move through such an angle that the end of the click shall travel through an arc slightly greater than some multiple of the pitch arc; the excess being simply to insure that the click shall clear the correct number of teeth on its return stroke, and have the smallest possible amount of lost motion. Thus, if the click and arm drive the wheel ahead two teeth at each forward stroke, the arc of motion of the click should be a trifle over twice the pitch arc, to insure the same amount of motion by each forward stroke.

204. Silent Click. — This contrivance (Fig. 146) avoids the clicking noise and the consequent wear of a common click. *BC* is the click, which, in this figure, is made to *push* the teeth.

Fig. 146



It is carried by one arm, *AB*, of a bell-crank lever, which has the same centre of motion, *A*, as the ratchet-wheel. The other arm of the lever has two studs, *E* and *E′*. Between these pins is the driving arm *AF*, also centred at *A*, and connected by a link *GH* to the click. The motion of this arm in the direction of the arrow drives the wheel in the same direction. When the motion of the arm is reversed, it at first moves back against *E′* before it can move the bell-crank lever; and, during this motion, the link *GH* lifts the click clear of the teeth. Then, by pressing against the pin *E′*, it moves the bell-crank back to the position shown in dotted lines. The driving arm then moves ahead again against the pin *E*, pulling the click into gear with the teeth, as shown; and then, by means of the pin *E*, drives the wheel ahead, as before.

205. Double-acting Click. — This arrangement, shown in Figs. 147 and 148, may be used when it is desired to drive the wheel ahead during *both* strokes of the driving arm.

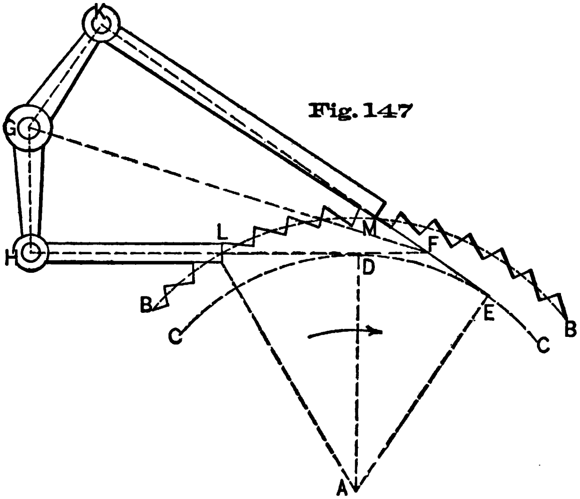


Fig. 147

To accomplish this result, the driving arm in Fig. 147 carries two *pushing* clicks; and that in Fig. 148, two *pulling* clicks. The former is the stronger arrangement, and is therefore used wherever great strength is required, as in ships' windlasses. The centre *G* being located, the arms *GK* and *GH* are made equal, and so placed that in their mid-positions they will be perpendicular to the lines of action of the respective clicks. With these arrangements, two or more detents or pawls are used, so placed that they will prevent the ratchet-wheel from turning back more than one-third or one-half the pitch.

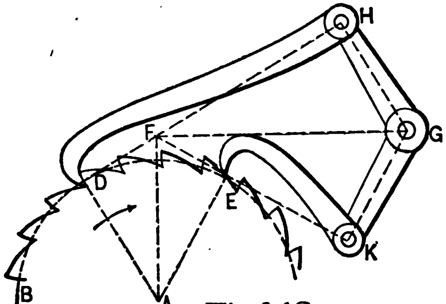


Fig. 148

206. Frictional Catch. — This contrivance is a sort of intermittent linkwork, founded on the dynamical principle, that two surfaces will not slide on each other so long as the angle which the direction of the pressure between them makes with their common normal at the point of contact is less than a certain angle, called the *angle of repose*. This angle depends on the material of which the surfaces are composed, their condition as to smoothness, and on the lubrication employed. For metallic surfaces, moderately smooth, not lubricated, the sine of this angle is somewhat greater than one-seventh. In Fig. 149, the shaft and rim of the wheel to be acted upon are shown in section. AK is the catch arm, having a rocking motion about the axis A of the wheel; the link by which it is driven is supposed to be jointed to it at K. $K'K''$ represents the stroke, or arc of motion, of the point K; so that $K'AK''$ is the angular stroke of the catch arm. L is a socket, capable of sliding up and down on the catch arm to a small extent; a shoulder for limiting the extent of that sliding is shown by dotted lines. The socket and the part of the arm on which it slides should be square, and not round, to prevent the socket from turning. From the side of the socket there projects a pin at *D*, from which the catch *DGH* hangs. *M* is a spring pressing against the forward side of the catch. *G* and *H* are two studs on the catch, which grip and carry forward the rim *BBCC* of the wheel during the forward stroke, by means of friction, but let it go during the return stroke.

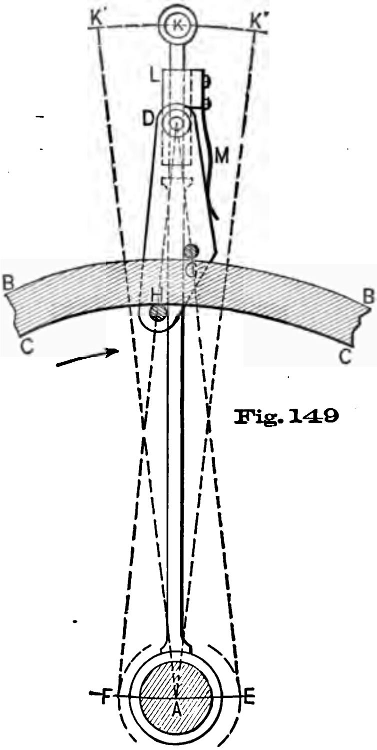


Fig. 149

A similar frictional catch, not shown, hanging from a socket on a fixed instead of a movable arm, serves for a detent, to hold the wheel still during the return stroke of the movable arm.

207. The following is the graphic construction for determining the proper position of the studs *G* and *H*: — Multiply the radii of the outer and inner surfaces, *BB* and *CC*, of the rim of the wheel, by a coefficient a little less than the sine of the angle of repose, — say one-seventh, — and with the lengths so found, as radii, describe two circular arcs about *A*; the greater (marked *E*) lying in the direction of forward motion, and the less (marked *F*) in the contrary direction. From *D*, the centre of the pin, draw *DE* and *DF* tangent to these arcs. Then *G*, where *DE* cuts *BB*, and *H*, where *DF* cuts *CC*, will be the proper positions for the points of contact of the two studs with the rim of the wheel.

The stiffness of the spring ought to be sufficient to bring the catch quickly into the holding position at the end of each return stroke.

The length of stroke of a frictional catch is arbitrary, and need not be an aliquot part of the circumference of the wheel, as is the case with the click motions described.

208. Another form of frictional catch is shown in Fig. 150.

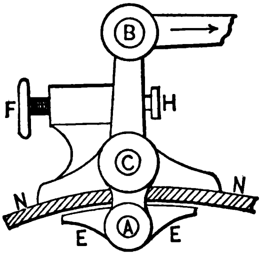


Fig. 150

An arm *AB*, centred at *C*, rides on a saddle which slides on the rim *NN* of the wheel. A piece *EE* is attached to one end of the arm, and admits of being pressed firmly against the inside of the rim *NN*. When the end *B* is moved as indicated, the rim *NN* will be firmly grasped or nipped between the saddle and the piece *EE*, and will be forced to move to the right. When *B* is pushed back, a stop prevents *BCA* from turning more than is sufficient to loosen the hold of *EE*, and the saddle slides freely on the rim. A screw *F* may be employed to bring up a stop *H* towards the arm *ACB*, and so to prevent the arm from twisting into the position which gives rise to the grip of *EE*. No motion will then be imparted to the wheel, a result which is obtained in any ordinary ratchet-wheel by throwing the click off the teeth.

CHAPTER XI.

COMMUNICATION OF MOTION BY WRAPPING CONNECTORS.

VELOCITY RATIO CONSTANT.

DIRECTIONAL RELATION CONSTANT.

Forms of Connectors and Pulleys. — Guide Pulleys. — Twisted Belts. Length of Belts.

209. It follows from Art. 27, that when the direction of the wrapping connector of two curves revolving in the same plane cuts the line of centres in a fixed point, the velocity ratio must be constant. The only curves used in practice are circles, the surfaces being surfaces of revolution rotating about fixed axes. In order that the motion may be continu-

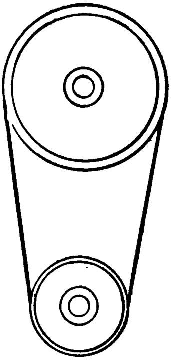


Fig. 151

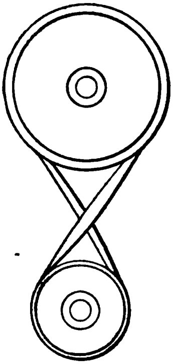


Fig. 152

ous, the ends of the wrapping connector are fastened together, forming an endless band which embraces a portion of the circumference of each wheel, or *pulley* as it is usually termed.

Where a direct or *open* band is used, as in Fig. 151, the direction of rotation of driver and follower is the same; but when the band is crossed, as in Fig. 152, the rotations take place in opposite directions.

210. Forms of Connectors and Pulleys. — Various materials are in use for wrapping connectors, the material depending to a certain extent upon the character and location of the machinery. The form of the pulleys depends largely upon the material of the connector. For very light machinery, such as sewing machines, the *bands* are usually round, and are made of leather, catgut, or woven cord. The pulleys used with such bands are grooved as illustrated by Fig. 153, the band running in this groove. For other machinery, where the distance between driver and follower is not very great, flat *belts* are used together with smooth pulleys. These pulleys are true cylinders in some cases, but are usually rounded to some extent as illustrated by Fig. 154. The amount of this convexity, or increase of radius from edge to centre of face, varies, according to different authorities, from nothing to one-half inch per foot of width of face of pulley. Average practice would seem to authorize one-eighth inch rise per foot of width.

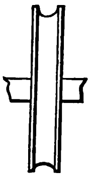


Fig. 153

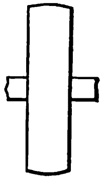


Fig. 154

Three or four such pulleys of different diameters are often made in one piece, the size of the pulleys increasing regularly from end to end. Such an arrangement is called a *stepped* pulley; and by means of two such pulleys, mounted on parallel shafts, and placed so that the smallest diameter of each is opposite the largest diameter of the other, motion can be transmitted between the shafts with a definite number of different velocity ratios. The various diameters must be so adjusted that the same length of belt (Art. 218) can be used in each case, the variation in the velocity ratio being obtained by simply transferring the belt from one pair of pulleys to another.

Flat belts are generally made of leather, either of a single thickness, or of two or more thicknesses sewed, riveted, or cemented together. The grain or hair side should be placed next the pulley. Woven cotton covered with vulcanized India rubber, and known as "rubber belting," is also largely used, particularly where dampness renders leather unfit. Paper and sheet iron have also been used to some extent.

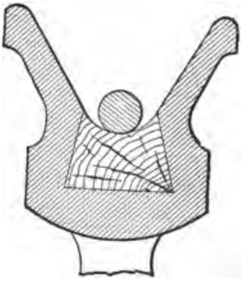


Fig. 155

For transmitting power over long distances, wire rope is used. The rims of the pulleys are grooved as shown in Fig. 155, the bottoms of the grooves being filled with wood, leather, oakum, or some other material, to reduce the wear of the wire rope.

For the running rigging of ships, tackles, and hoisting machinery, hemp or other similar rope is used with smooth grooved pulleys similar to the one shown in Fig. 153.

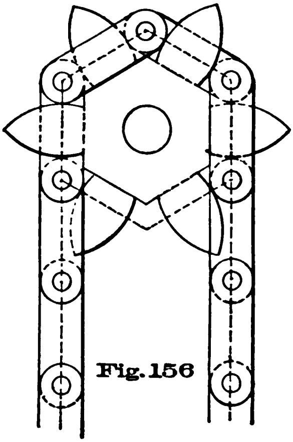


Fig. 156

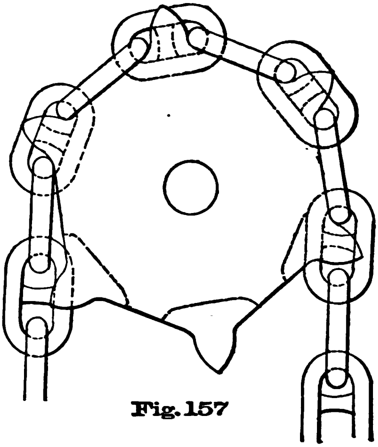


Fig. 157

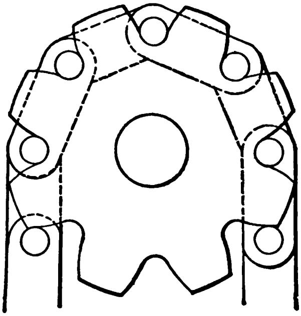


Fig. 158

Where great strength is required in small compass, iron chains are used. The links of the chains are of various shapes. The pulleys are formed to fit the links more or less nearly, or with teeth to enter the links, and thus prevent slipping. Figs. 156, 157, and 158 illustrate forms of chains and pulleys.

211. Tightening Pulleys.—When smooth pulleys are used, the motion is transmitted directly by the friction between the belts or bands and the pulleys. Ordinarily the tension of a belt, properly fitted, is sufficient to produce the necessary adhesion. But in some cases, tightening pulleys are employed to prevent slipping; as, for example, they are frequently employed for this purpose on the driving belts of stationary steam-engines. These tightening pulleys are pressed against the belt by weights or springs, and thus maintain a constant tension, or are mounted in a frame which can be adjusted in position by screws.

212. Shifting Belts.—A flat belt may be easily shifted from one position on a cylindrical pulley to another position by pressing the belt in the required direction on the *advancing* side, while pressure on the *retreating* side will produce no effect. Thus, if we press a belt in the above manner as shown in Fig. 159, it is clear, that, as the pulley continues to revolve, the successive portions of the belt come into contact with the pulley at points to the left of the original position, and as the revolution of the pulley carries them in a direction perpendicular to the axis, the position of the belt on the pulley is gradually changed. If we had pressed the belt on that part which had left the pulley, its position on the pulley would not have been affected.

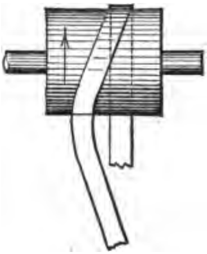


Fig. 159

From the above it follows, that, if the central line of the advancing side of a flat uniform belt is kept in the central plane of the pulley, it will run true without any tendency to leave the pulley.

213. Convexity of Pulley. — If we place a flat belt on a convex pulley, as shown in Fig. 160, the tension at the edge *D* will evidently be greater than at the edge *F*. Consequently the tendency will be to throw the belt into the position shown dotted. If we now rotate the pulley, the belt will, as shown in the preceding article, be moved to the left or towards the largest diameter of the pulley. It is for this reason that the convexity is given to pulleys, so that if for any reason the belt commences to come off, the increased tension of one part will bring it back to a central position.

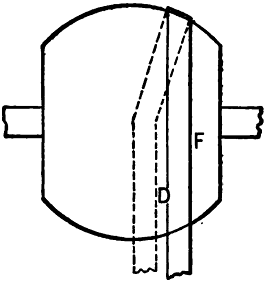


Fig. 161

214. Twisted Belt. — In Fig. 161, let *A* be a fixed axis carrying the pulley *D*, and let *B* be an axis carrying the pulley *E*. At first consider the axis *B* and pulley *E* to occupy the position shown dotted, so that *A* and *B* are parallel, and *D* and *E* are in the same plane. Let *SS* be the common tangent to the two pulleys, drawn on the sides from which the belt is delivered, and in the central planes of the pulleys. Now, let axis *B* and pulley *E* be turned about *SS* into some other position such as that shown in full lines. Then *SS* will be the intersection of the central planes of the pulley. If, now, the pulleys be rotated as indicated by the arrows, it follows, that since the points *a* and *b* are in the central plane of *E*, and *a′* and *b′* are in the central plane of *D*, the *advancing* side of the belt is in each case in the central plane of the pulley considered, and the belt will not tend to run off (Art. 212). But if the pulleys be rotated in the opposite direction, the belt will immediately run off. Hence this arrangement can only be used when the axes are always to revolve in the same direction. In laying out twist-belt motions, the circles *D* and *B* should be taken equal to the largest diameter of the respective pulleys plus the thickness of the belt.

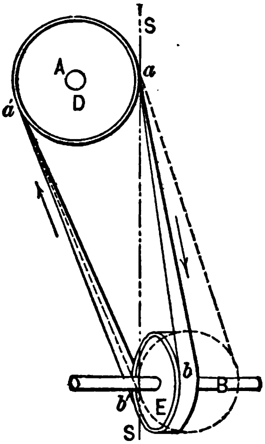


Fig. 161

215. Guide Pulleys are used to change the direction of belts. In Fig. 162, let *ca* be the direction of a belt which we wish to change to the direction *ab*. If we place a fixed pulley of any convenient diameter in the angle *cab* with its axis on the line bisecting this angle, and so that the lines *ca* and *ab* are both tangents to the pulley, it is evident that by means of this pulley the desired change may be effected. If the directions do not intersect within a convenient distance, as in Fig. 163, connect them by a line *de* at any convenient points, and place guide pulleys in the angles at *d* and *e*, as shown.

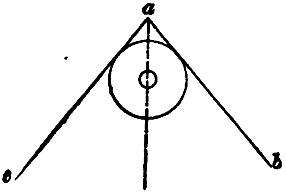


Fig. 162

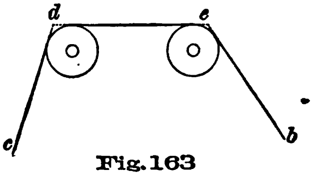


Fig. 163

216. By means of guide pulleys we can connect axes neither parallel nor meeting in direction, so that they may be rotated in either direction. In Fig. 164, *SS* is the intersection of the central planes of the pulleys *A* and *B*. Assume

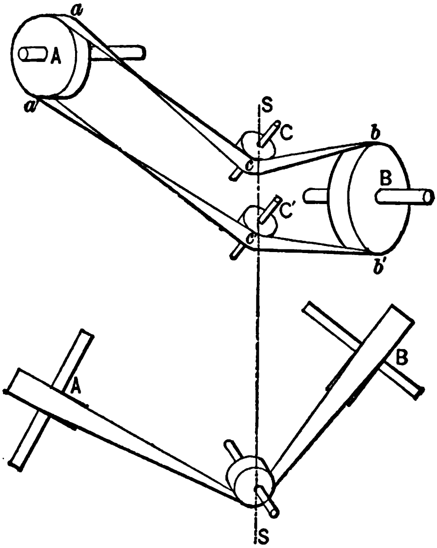


Fig. 164

points c and c' in this line, and draw tangents to the pulleys A and B. Then the guide pulleys C and C' should evidently be placed in the planes of these tangents, and so as to be tangent to ca, cb, and $c'a'$, $c'b'$, respectively. By this arrangement the direction of the belt where it leaves a pulley is always in the central plane of the next pulley, and hence the belt can be run in either direction without tending to leave the pulleys.

217. In Figs. 165 and 166 are shown applications of guide pulleys, the rotation being always in the same direction. In Fig. 165, two axes which lie in the same plane and make a small angle with each other are connected, so as to be capable of rotation in the direction of the arrow. One guide pulley is used, and the arrangement depends on the principles of Art. 214.

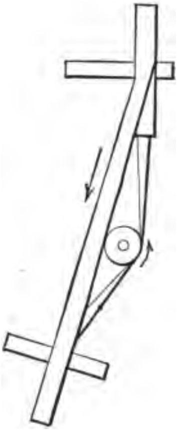


Fig. 165

In Fig. 166, two pulleys on parallel shafts, but not in the same plane, are connected, so as to be capable of rotation in one direction by means of two guide pulleys fixed on the same shaft. The diameter of the guide pulleys should be equal to the distance between the central planes of the driving and following pulleys.

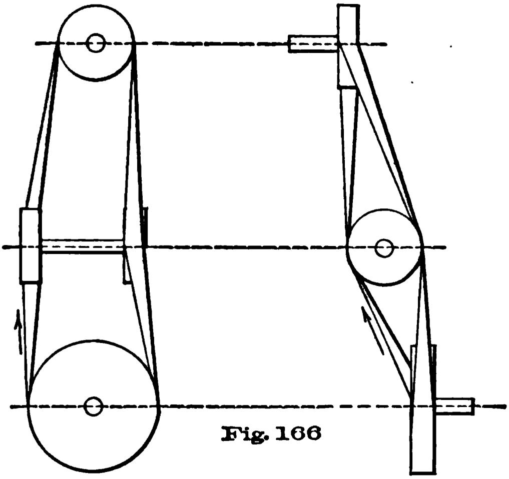


Fig. 166

218. Length of Belt. — The actual length of belt required in every case is best determined by actual measurement over the pulleys, or by measurement on a scale drawing. It may, however, be calculated in the following manner. In Figs. 167 and 168, let *A* and *B* be the axes of two pulleys

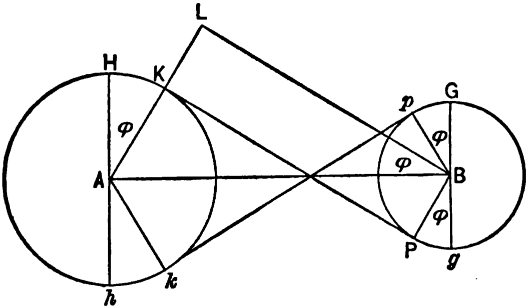


Fig. 167

connected by belts, KP and kp being the straight parts of the belt. Draw hAH and gBG perpendicular to AB; AK and Ak, BP and Bp, radii of the pulleys to the points of tangency of the belt; and BL parallel to KP. Let $HAK = PBg = ABL = \phi$; $AB = d$; $AK = R$; $BP = r$. Then, for a *crossed* belt (Fig. 167),

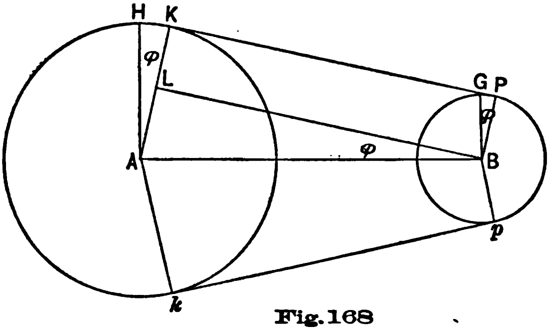


Fig. 168

$$KP = \sqrt{d^2 - (R + r)^2}.$$

Arc of contact for pulley, radius R

$$= R(\pi + 2\phi) = R\left(\pi + 2 \operatorname{arc} \sin \frac{R + r}{d}\right).$$

Arc of contact for pulley, radius r

$$= r\left(\pi + 2 \operatorname{arc} \sin \frac{R + r}{d}\right).$$

$\therefore$ Total length of *crossed* belt $= L$

$$= 2\sqrt{d^2 - (R + r)^2} + (R + r)\left(\pi + 2 \operatorname{arc} \sin \frac{R + r}{d}\right).$$

For an *open* belt (Fig. 168),

$$KP = \sqrt{d^2 - (R - r)^2}.$$

Arc of contact for pulley, radius R

$$= R\left(\pi + 2 \operatorname{arc} \sin \frac{R - r}{d}\right).$$

Arc of contact for pulley, radius r

$$= r\left(\pi - 2 \operatorname{arc} \sin \frac{R - r}{d}\right).$$

$$\therefore \quad \text{Total length of } \textit{open} \text{ belt} = L = 2\sqrt{d^2 - (R - r)^2} + \pi(R + r) + 2(R - r) \operatorname{arc} \sin \frac{R - r}{d}.$$

It is to be noted, that, for a given value of d, the length of a crossed belt depends upon the *sum* of the radii of the pulleys, while the length of an open belt depends both upon the sum and difference of the radii. It follows from this, that one *crossed* belt can be used to transmit different velocity ratios between two shafts, with the single condition that the sum of the radii of each pair of pulleys must be the same. For example: with any given value of d, the same belt, crossed, will exactly fit pulleys having diameters of 4 and 16, 6 and 14, 8 and 12, 10 and 10; the value of $R + r$ in each case being 10. But these pulleys could not be exactly fitted with the same length of *open* belt.

219. Approximate Formulæ. — As the above exact formulæ are cumbersome, the following approximate equations are introduced. All dimensions are best taken in *inches*, and the signification of the letters is the same as above. The formulæ will give results which are safe within the prescribed limits.

For crossed belt, $L = 3\frac{5}{8}(R + r) + 2d$.

To be used when $\frac{R + r}{d}$ does not exceed 0.23.

For open belt, $L = 3\frac{1}{4}(R + r) + 2d$.

To be used when $\frac{R - r}{d}$ does not exceed 0.16.

Within these limits the results are a trifle large, while beyond them they fall short.

220. Wrapping connectors may be used to transmit motion when the directional relation or the velocity ratio, or both, are variable. The result is obtained by using non-circular pulleys, or by winding the band in a spiral groove of variable radius.

In such cases the length of the band is not usually constant, and tightening pulleys must usually be employed to insure the requisite tension.

In practice, variable conditions are so much better met by other modes of connection, that wrapping connectors are scarcely ever used for this purpose; and hence no discussion of such use is here given.

CHAPTER XII.

TRAINS OF MECHANISM.

Value of a Train. — Directional Relation in Trains. — Clockwork. — Notation. — Method of designing Trains. — Approximate Numbers for Trains.

221. The required velocity ratio of two motions being given, it is always *theoretically* possible to obtain this ratio by the use of one of the *elementary combinations* described in the previous chapters. It often happens, however, that this ratio is so small, or so large, that, *practically*, the motion is better communicated by a train of such combinations; each piece being at the same time the follower of the piece that drives it, and the driver of the piece that follows it.

For convenience, let us first consider the case in which all the pieces are circular wheels revolving about fixed axes. The usual arrangement in such cases is to secure two unequal wheels upon each axis, except the first and last, and to make the larger wheel on each axis gear with the smaller wheel on the next axis.

222. Value of a Train. — Let there be m such axes, and let us designate by ϵ the value of the train; that is, the ratio of the angular velocities of the first and last axes, or, what amounts to the same thing, the ratio of their synchronal rotations.

Let a_1, a_2, a_3, . . . a_m be the angular velocities of the successive axes. Then we have

$$\epsilon = \frac{a_m}{a_1} = \frac{a_2}{a_1} \times \frac{a_3}{a_2} \times \frac{a_4}{a_3} \times \dots \frac{a_m}{a_{m-1}}. \qquad (1)$$

That is, the value of the train may be found by multiplying together the separate ratios of the angular velocities of the successive pairs of axes.

Again, let the synchronal rotations of the successive axes of the train be L_1, L_2, L_3, . . . L_m. Then we have

$$\epsilon = \frac{L_m}{L_1} = \frac{L_2}{L_1} \times \frac{L_3}{L_2} \times \frac{L_4}{L_3} \times \dots \frac{L_m}{L_{m-1}}. \qquad (2)$$

That is, the value of the train may be found by multiplying together the separate ratios of the synchronal rotations of the successive pairs of axes. The value of ϵ is, of course, the same in both the above equations. This value will not be affected by the substitution, for any of the intermediate ratios, of any other two numbers that are in the same proportion; hence we may express the values of those ratios in the terms that may most easily be obtained from the train whose motions we wish to consider.

Letting a = angular velocity of one of two wheels in gear, R its radius, N its number of teeth, P its period, or time of one rotation, and L its number of rotations in a given time; and letting a', R', N', P', and L' be the corresponding quantities for the other wheel, we have (Art. 35),

$$\frac{a'}{a} = \frac{R}{R'} = \frac{N}{N'} = \frac{P}{P'} = \frac{L'}{L}, \qquad (3)$$

which equation will enable us to write the proper ratio in each case.

For instance, let N_1, N_2, N_3, . . . N_{m-1} be the numbers of teeth of the drivers on the successive axes, and let n_2, n_3, n_4, . . . n_m be the numbers of teeth of the corresponding followers. Then we may write, by making the proper substitutions for the intermediate ratios in Equation (1),

$$\epsilon = \frac{a_m}{a_1} = \frac{L_m}{L_1} = \frac{N_1}{n_2} \times \frac{N_2}{n_3} \times \frac{N_3}{n_4} \times \ldots \frac{N_{m-1}}{n_m}$$

$$= \frac{N_1 \times N_2 \times N_3 \times \ldots N_{m-1}}{n_2 \times n_3 \times n_4 \times \ldots n_m}. \qquad (4)$$

That is, the value of the train is equal to the quotient obtained by dividing the continued product of the numbers of teeth of all the drivers by the continued product of the numbers of teeth of all the followers.

It is obvious, that, in a train of this kind, the number of drivers, as well as the number of followers, is always one less than the whole number of axes.

223. Practical Example. — It is not necessary that *all* the ratios should be expressed in the *same* terms. As before stated, it is simply necessary to use, for each ratio, two numbers in the proper proportion.

For example, let there be a train of six axes, connected as above described.

Let the first axis revolve once per minute, and let the second axis revolve once in fifteen seconds. Hence

$$\frac{P_1}{P_2} = \frac{60}{15}.$$

Let the second axis revolve three times while the third revolves five times. Hence

$$\frac{L_3}{L_2} = \frac{5}{3}.$$

Let the third axis carry a wheel of sixty teeth, driving a wheel of twenty-four teeth on the fourth axis. Hence

$$\frac{N_3}{n_4} = \frac{60}{24}.$$

Let the fourth axis carry a pulley of twenty-four inches diameter, driving, by means of a belt, a pulley of twelve inches diameter on the fifth axis. Hence

$$\frac{R_4}{R_5} = \frac{24}{12}.$$

Let the fifth axis turn with an angular velocity two-thirds as great as that of the sixth axis. Hence

$$\frac{a_6}{a_5} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

Substituting these ratios for the successive terms of Equation (1), we get

$$\epsilon = \frac{a_6}{a_1} = \frac{P_1}{P_2} \times \frac{L_3}{L_2} \times \frac{N_3}{n_4} \times \frac{R_4}{R_5} \times \frac{a_6}{a_5}$$
$$= \frac{60}{15} \times \frac{5}{3} \times \frac{60}{24} \times \frac{24}{12} \times \frac{3}{2} = 50.$$

That is, the angular velocity of the last axis is fifty times as great as that of the first; in other words, the last axis will make fifty revolutions in the same time that the first axis revolves once.

224. Directional Relation in Trains. — In this manner, we may find the synchronal rotations of the extreme axes in any train of mechanism. Their directional relation depends on the number, and the manner of connection, of the axes. In a train consisting solely of spur wheels or pinions on fixed parallel axes, the direction of rotation of the successive axes will be alternately in opposite directions.

Hence, if the train consists of an *odd* number of axes, the first and last axes will revolve in the *same* direction; if it consists of an *even* number of axes, they will revolve in *opposite* directions.

In this connection, it must be remembered that an annular wheel (Art. 36) revolves in the same direction as its pinion.

When the axes in a train are not parallel, the directional relation of the extreme axes can only be ascertained by tracing the separate directional relations of each successive pair of axes in order.

Two separate wheels in a train may revolve concentrically about the same axis; as, for example, the wheels to which are attached the hands of a clock. In this case, one of the wheels is fixed on the axis as usual, and the other is fixed on a tube, or *cannon* as it is sometimes called, which revolves freely on the first axis.

If these wheels are to move in opposite directions, a single bevel wheel may be used to connect them; but if they are to turn in the same direction, as in a clock, they must be made in the form of spur-wheels, and connected by means of two other spur-wheels fixed to an axis parallel to the first.

225. Idle Wheel. — Let a spur-wheel be placed between and in gear with two other spur-wheels. Let the radii of the first, middle, and last wheels be R_1, R_2, R_3, and let their angular velocities be a_1, a_2, a_3. Then we have, for the first and middle wheels,

$$\frac{a_2}{a_1} = \frac{R_1}{R_2},$$

and, for the middle and last wheels,

$$\frac{a_3}{a_2} = \frac{R_2}{R_3}.$$

Multiplying these equations together, we get

$$\frac{a_3}{a_1} = \frac{R_1}{R_3}.$$

That is, the velocity ratio of the two extreme wheels is precisely the same as though they were in immediate contact. The intermediate wheel is called an *idle wheel;* and, though it does not affect the velocity ratio, it does affect the directional relation. For, if the two extreme wheels were in direct contact, they would revolve in opposite directions; but, by the introduction of the idle wheel, they are caused to revolve in the same direction.

226. Clockwork. — A familiar example of the employment of a train of wheels is afforded by a common clock.

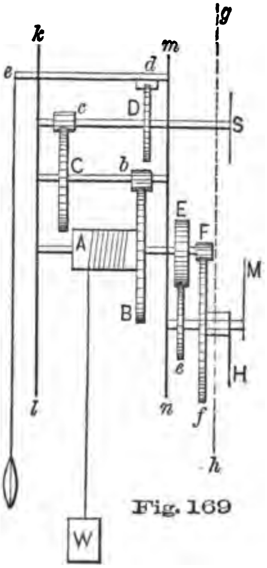


Fig. 169

In Fig. 169 is shown the arrangement of the wheels in a clock of the simplest kind. *A* is the *barrel*, and around it is wound a cord to the end of which is fastened the weight *W*. On the same axis with *A* is fixed the spur wheel *B*, which gears with the pinion *b* on a second axis. On the latter is also fixed the spur wheel C, gearing with the pinion c on the third axis. This axis also carries an escapement wheel D (Art. 180), the verge or anchor d being fixed to the fourth axis, to which the pendulum is also hung at e. One tooth of the escape-wheel crosses the line of centres for every two vibrations of the pendulum. Let the time of one vibration of the pendulum be t seconds, and let the escape-wheel have Δ teeth; then the period or time of one complete rotation of this wheel is $2t\Delta$ seconds. To take a simple case, let the pendulum be a seconds pendulum; then $t = 1$, and if $\Delta = 30$, the swing-wheel will make one complete revolution in $2t\Delta = 2 \times 30 = 60$ seconds $= 1$ minute. Let B have 48 teeth; b, 6 teeth; C, 45 teeth; and c, 6 teeth. Then we have, for the value of the train connecting the barrel axis and the escapement axis,

$$\epsilon = \frac{a_3}{a_1} = \frac{L_3}{L_1} = \frac{48 \times 45}{6 \times 6} = 60.$$

That is, the escapement axis (or *arbor*, using the term employed by clockmakers) will make sixty revolutions while the barrel arbor makes one. Hence the barrel arbor will revolve once in sixty minutes, or one hour. The barrel A is not permanently secured to this arbor, but is connected to it, or to the wheel B, by means of a click and ratchet (Art. 201); so that, while it is free to move in one direction, its rotation in the other direction compels the wheel B to rotate with it. This arrangement permits the barrel to be rotated so as to wind up the cord without affecting the rest of the train. The number of times that the cord is wound round the barrel evidently depends on the length of time that the clock is to run without being wound. Generally not over sixteen coils of cord are so employed, which, in our clock, as the barrel arbor revolves once an hour, would be sufficient to make the clock run sixteen hours without re-winding.

227. The train of wheel-work just described is solely destined for the purpose of communicating the action of the weight to the pendulum in such a manner as to supply the loss of motion from friction and the resistance of the air. But besides this, the clock is required to indicate the hours and minutes by the rotation of two separate hands, and accordingly two other trains of wheel-work are employed for this purpose. The train just described is generally contained in a frame, consisting of two plates, shown edgewise at *kl*, *mn*, which are kept parallel and at the proper distance by means of three or four pillars not shown in the diagram. Opposite holes are drilled in these plates, which receive the pivots of the axes or arbors already described. But the axis which carries *A* and *B* projects through the plate, and other wheels *E* and *F* are fixed to it. Below this axis, and parallel to it, a stout pin or *stud* is fixed to the plate. On this stud revolves a tube, to one end of which is fixed the minute-hand *M*, and to the other the wheel *e* in gear with *E*. In our present clock, the wheel *E*, being fixed to the barrel arbor, revolves once an hour; and as the minute-hand must also revolve once in that period, the wheel *E* and *e* must be equal. A second and shorter tube is fitted upon the tube of the minute-hand so as to revolve freely, and this carries at one end the hour-hand *H*, and at the other a wheel, *f*, which is driven by the pinion *F*. As *f* must revolve once in twelve hours, it must have twelve times as many teeth as *F*.

228. Notation. — In discussing problems concerning trains of mechanism, we soon feel the need of some scheme of notation, whereby we may show, clearly and concisely, all the facts concerning the train which affect the transmission of motion. It is desirable to show, primarily, the order and nature of the several parts, and the manner in which the motion is transmitted; but such a scheme should also admit of the addition of dimensions and nomenclature, and should afford a ready means of calculating the velocity ratio.

Let the wheels be represented by their numbers of teeth, and write these numbers, beginning with the first driver, in horizontal lines; all the wheels that are on the same axis having their numbers written on the same horizontal line, and all the wheels that are in gear having the numbers of the followers written vertically below those of the respective drivers.

229. Example. — Thus, in the principal train of the clock (Fig. 169), if the letters represent the wheels, we should write the train thus: —

$$\begin{array}{lll} B & & \\ b — C & & \\ & c — D; & \end{array}$$

or, employing the numbers already selected,

$$\begin{array}{lll} 48 & & \\ 6 — 45 & & \\ & 6 — 30 & \end{array}$$

Similarly we may represent the whole mechanism of our clock, adding to the numbers the names wherever it may be thought necessary. Thus —

Barrel 48 ——————————— 25 ——— 4
6 — 45
6 — 30 swing-wheel.
25 minute-hand.
48 hour-hand.

The above shows clearly the three trains of mechanism from the barrel to the swing-wheel, the minute-hand, and the hour-hand respectively. It also distinctly classifies the pieces as drivers or followers, as the case may be, and shows the nature of their connection; that is, whether they are permanently fixed to the same axis, or connected by gearing. In case other connections are employed, such as links or bands, this must be written in the diagram, or expressed by a proper sign.

230. Method of Designing Trains. — We are now ready to undertake the solution of a problem of considerable importance in the contrivance of mechanism; namely, Given the velocity ratio of the extreme axes or pieces of a train, to determine the number of intermediate axes, and the proportions of the wheels, or numbers of their teeth. For simplicity, we will suppose the train to consist of toothed wheels only; for a mixed train, consisting of wheels, pulleys, link-work, and sliding-pieces, can be calculated upon the same principles. Let the synchronal rotations of the first and last axes of the train be L_1 and L_m respectively, and let N_1, N_2, . . . etc., be the numbers of teeth in the drivers, and n_2, n_3, etc., the number of teeth in the followers; then the value of the train is

$$\epsilon = \frac{L_m}{L_1} = \frac{N_1 \times N_2 \times N_3 \times \ldots N_{m-1}}{n_2 \times n_3 \times n_4 \times \ldots n_m},$$

both numerator and denominator of this fraction being composed of $m - 1$ terms.

The value of ϵ being given in this shape, an equal fraction must be found, whose numerator and denominator shall each admit of being divided into $m - 1$ factors of convenient magnitude for the number of teeth of a wheel.

The value of m, that is, the number of axes, is sometimes given with the other data of the problem, but more usually it is one of the quantities that are to be determined.

The order of succession of the drivers and followers is a matter of indifference, so far as the velocity ratio is concerned; for the value of the above fraction will evidently not vary with any variation in the order of the factors of either the numerator or denominator.

231. Least Number of Axes. — The number of axes will evidently depend upon the limits between which the numbers of teeth are to be allowed to vary.

For instance, let w be the *greatest* number of teeth that can be conveniently assigned to a wheel, and let p be the *least* that can be given to a pinion. Now, in any given case, let us suppose L_m greater than L_1, so that the wheels will be the drivers, and the pinions the followers. The least number of axes will then evidently be obtained by giving each wheel w teeth, and each pinion p teeth. The number of axes being m, we will have (Art. 222) $m - 1$ wheels and $m - 1$ pinions. Hence

$$\epsilon = \frac{L_m}{L_1} = \frac{N_1 \times N_2 \times N_3 \times \ldots N_{m-1}}{n_2 \times n_3 \times n_4 \times \ldots n_m}$$

$$= \frac{w \times w \times w \text{ to } (m-1) \text{ factors}}{p \times p \times p \text{ to } (m-1) \text{ factors}} = \left(\frac{w}{p}\right)^{m-1}; \quad (5)$$

whence $\log \epsilon = (m - 1)(\log w - \log p)$,

$$\therefore \quad m = 1 + \frac{\log \epsilon}{\log w - \log p}. \qquad (6)$$

The least number of axes, under the assigned conditions of w and p, is evidently the value of m thus found, if this value is a whole number; or the whole number next larger than this value of m, if the latter is fractional. No general rule can be given for determining the values of w and p, which are governed by considerations that vary according to the nature of the proposed machine; also, it will rarely happen that the fraction will admit of being divided into factors so nearly equal as to limit the number of axes to the smallest value so assigned.

232. Practical Example of Clock Train. — We will now return to the consideration of the clock described in Art. 226, and show how the number of axes and the number of teeth of the wheels and pinions were determined. It was required that the first or barrel axis should revolve once per hour, and that the m^{th} or swing-wheel axis should carry a seconds hand, S. The swing-wheel axis must therefore revolve once per minute, or sixty times per hour.

Consequently

$$\epsilon = \frac{L_m}{L_1} = \frac{60}{1} = \frac{N_1 \times N_2 \times \ldots N_{m-1}}{n_2 \times n_3 \times \ldots n_m}.$$

Let D be the numerator of this fraction, i.e., the continued product of all the drivers, and let F be the denominator, i.e., the continued product of all the followers.

Then

$$\epsilon = 60 = \frac{D}{F} \quad \therefore \quad D = 60 \times F,$$

an indeterminate equation, for the solution of which any numbers may be employed that are within the assigned limits of w and p. Now, in ordinary clocks, $w = 60$, and $p = 6$, so that

$$\frac{w}{p} = \frac{60}{6} = 10.$$

From Equation (5), we have

$$\epsilon = 60 = (10)^{m-1}.$$

We can then determine the value of m by means of Equation (6); or, what is much simpler, determine, by inspection, the value of $m - 1$ from the above expression. The latter method is to be preferred, as the *exact* value of $m - 1$, if it be fractional, is of no consequence, it being simply necessary to determine the next greater whole number.

Thus, in our example, it is evident, as 60 lies between 10^1 and 10^2, that $m - 1$ must lie between 1 and 2, consequently m must lie between 2 and 3; and, taking the next larger whole number, we fix on $m = 3$, as the least number of axes. Consequently there will be two wheels and two pinions. Taking the pinions at six teeth each, we have

$$\epsilon = 60 = \frac{D}{F} = \frac{D}{6 \times 6}$$

$$\therefore \quad D = 60 \times 6 \times 6 = 2160,$$

which is the product of the two wheels.

We are at liberty to divide this into any two suitable factors. The best mode of doing it is to begin by dividing the number into its prime factors, and writing it in this form,

$$2160 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5.$$

For this enables us to see clearly the composition of the number, and it is easy to distribute these factors into two groups; as, for example,

$$(2 \times 2 \times 2 \times 2 \times 3) \times (3 \times 3 \times 5) = 48 \times 45,$$

or

$$(2 \times 2 \times 2 \times 5) \times (2 \times 3 \times 3 \times 3) = 40 \times 54,$$

or

$$(2 \times 2 \times 3 \times 3) \times (2 \times 2 \times 3 \times 5) = 36 \times 60.$$

The first group will give us the two wheels that are most nearly equal, which is a sufficient reason for selecting that pair for our train. We now have

$$\frac{D}{F} = \frac{48 \times 45}{6 \times 6}.$$

So far we have only determined on the numbers of the teeth of the various wheels, without locating them as regards the different axes; and the above fractional expression is an excellent method of exhibiting the train under these conditions.

As before stated, the order in which the wheels come is a matter of indifference, so far as the velocity ratio is concerned; and, as no other considerations enter into this case, we will place driver 48 on the first axis, follower 6 and driver 45 on the second axis, follower 6 and swing-wheel 30 on the third axis, giving us, as in Art. 229, the train

$$\begin{array}{lll} 48 & & \\ 6 \text{ — } 45 & & \\ & 6 \text{ — } 30. & \end{array}$$

233. Another Clock Train. — Six is, however, too small a number of leaves for the pinion, if perfect action is desired; for it is evident, from the table of Art. 134, that a pinion of 6 teeth cannot drive a wheel of less than 21 teeth, if the arc of recess equal two-thirds pitch; while, if this arc is increased to three-fourths pitch, a pinion of 6 cannot be made to work at all. In well-made clocks, p is generally taken between 8 and 12, while w ranges from 100 to 120.

Let us find a new train for our clock, having $p = 12$, and $w = 105$.

We have

$$\epsilon = 60 = \left(\tfrac{105}{12}\right)^{m-1} = (8.75)^{m-}$$

from which we see, by inspection, that the fractional, and lies between 1 and 2; that the value of m lies between 2 and 3; and that the least number of axes will consequently be 3. Assuming the two pinions to be equal, and to have the smallest allowable number of teeth, we have

$$\frac{D}{F} = \frac{N_1 \times N_2}{12 \times 12} = 60 \qquad \therefore \quad D = 60 \times 12 \times 12 = 8640.$$

Proceeding as in the last example, we find the best values for the wheels to be $D = 96 \times 90$. We then have

$$\frac{D}{F} = \frac{96 \times 90}{12 \times 12},$$

and, placing them on their axes, we have the train

96
12 — 90
12.

Instead of assuming the pinions, we might have started with the wheels. Thus let us take

$$\frac{D}{F} = \frac{105 \times 105}{n_2 \times n_3} = 60$$

$$\therefore \quad F = \frac{105 \times 105}{60} = 183.75.$$

It is evidently impossible to divide 183.75 into two *integer* factors; and, as we cannot increase the assumed number of teeth for the wheels, we must diminish the number of one or both. Let us take one of the wheels as 104. This will give us

$$F = \frac{104 \times 105}{60} = 182,$$

which can readily be factored, giving us $F = 13 \times 14$, and the train

$$\begin{array}{lll} 104 & & \\ 13 & — 105 & \\ & & 14. \end{array}$$

It very often happens, as just illustrated, that attempting to make the wheels and pinions with the limiting numbers of teeth gives rise to very awkward results, while an excellent train can, in such cases, be generally found by trying several numbers *within* the limits.

234. Clock with rapidly vibrating Pendulum. — If a clock has no seconds hand, the limitation as to the period of one revolution of the swing-wheel axis is removed. This is an advantage in clocks having short, and consequently rapidly vibrating, pendulums; for it would be impracticable to make the period of the swing-wheel axis one minute, as before, on account of the great number of teeth which would be required for the swing-wheel. If t = time of vibration of the pendulum in seconds, and Δ = number of teeth in the swing-wheel, then (as in Art. 226) $2t\Delta$ is the time required for one revolution of the swing-wheel.

But the vibrations of short pendulums are commonly expressed by stating the number of them in a minute. Let S be this number; then $\frac{2\Delta}{S}$ is the time of one revolution of the swing-wheel in minutes; $\frac{S}{2\Delta}$ is the number of revolutions of the swing-wheel axis per minute; and, as the barrel arbor revolves once per hour, we have for the train between them,

$$\epsilon = \frac{D}{F} = \frac{60S}{2\Delta} = \frac{30S}{\Delta}.$$

For example, let the pendulum of a clock make 170 vibrations per minute; let there be 25 teeth on the swing-wheel; then

$$\epsilon = \frac{D}{F} = \frac{30 \times 170}{25} = 204.$$

Taking $w = 128$, and $p = 8$, we have

$$\frac{w}{p} = \frac{128}{8} = 16;$$

and, as $204 = (16)^{m-1}$, we see, by inspection, that the least number of axes is 3.

Assuming the pinions as each having 8 teeth, we have

$$D = 204 \times F = 204 \times 8 \times 8 = 13056 = 128 \times 102.$$

Hence the train is

$$\begin{array}{llll} 128 & & & \\ 8 & - 102 & & \\ & & 8 & - 25. \end{array}$$

235. Eight-Day Clock. — All the trains so far explained were designed to establish the proper velocity ratio between the hour arbor and the swing-wheel axis. It was assumed in each case that the hour arbor also carried the weight-barrel; and, as we limited the number of coils of the cord to 16, it follows that the clocks so far considered will only run 16 hours without re-winding.

If we adhere to the limitation as to the number of coils of the cord, but still desire the clock to run longer than 16 hours, the barrel must be attached to a separate axis connected by wheel-work with the hour arbor, so that the barrel may revolve more slowly, consequently taking more time to uncoil all the cord.

For example, let the clock be required to go 8 days without re-winding; then, with 16 coils of cord on the barrel, the latter must revolve once in $\frac{8 \times 24}{16} = 12$ hours. Then, assuming $w = 100$, and $p = 8$, we may use the train, —

		Periods.
Barrel arbor, 96		12 hours.
Hour arbor	8 — 90	1 hour.
	12 — 96	8 minutes.
Minute arbor	12 — 30, swing-wheel	1 minute.

It is often convenient to add to the notation the periods of the different arbors, as has been done in this case.

236. Month Clock. — Let the clock be required to run 32 days without re-winding, and let there be 16 coils on the barrel as before; then the latter must revolve once in $\frac{32 \times 24}{16} = 48$ hours. The train from the barrel to the hour arbor is $\frac{D}{F} = 48$, which will require an intermediate axis.

Letting $w = 100$, and $p = 12$, we may employ the following train: —

		Periods.
Barrel arbor, 96		48 hours.
	16 — 96	8 hours.
Hour arbor	12 — 90	1 hour.
	12 — 96	8 minutes.
Minute arbor	12 — 30, swing-wheel,	1 minute.

237. Now, in the clock (Fig. 169), the arbor of A is made to revolve in one hour, because the wheels E and e are equal. By making these wheels of different numbers, we get rid of the necessity of providing, in the principal train, an arbor that shall revolve in one hour; and we may thus, in many cases, distribute the wheels more equally. For example, in an eight-day clock let the swing-wheel revolve once per minute, and let the train from the barrel-arbor to this minute-arbor be

$$\frac{D}{F} = \frac{108 \times 108 \times 100}{12 \times 12 \times 10} = 810,$$

in which case the barrel will revolve once in 810 minutes, or $13\frac{1}{2}$ hours.

The second wheel of this train, which, in Fig. 169, corresponds to D, will revolve in $\frac{12}{108} \times 810 = 90$ minutes, or $1\frac{1}{2}$ hours. On its arbor must be fixed, as in the figure, the wheels E and F for the minute and hour hands; and we may employ, for the two pairs of wheels,

$$\frac{F}{f} = \frac{1\frac{1}{2}}{12} = \frac{1}{8} = \frac{10}{80}; \quad \text{and} \quad \frac{E}{e} = \frac{1\frac{1}{2}}{1} = \frac{3}{2} = \frac{54}{36}.$$

So that our train will be as follows:—

			Periods.
Barrel, 108			810 minutes.
12 — 108	54	10	90 "
12 — 100			10 "
10 — 30 (swing-wheel)			1 minute.
	36 (minute-hand)		1 hour.
		80 (hour-hand)	12 hours.

238. The above examples have been confined to clock-work, because the action is more generally understood than that of other machines. The principles and methods are, however, universally applicable, or, at least, require very slight modifications to adapt them to particular cases.

For instance, in a screw-cutting lathe, there is usually one intermediate axis between the leading-screw and the head-stock spindle. Let the leading-screw be right-handed, and have two threads to the inch; let $w = 130$, $p = 20$; and let it be required to cut a right-handed screw of 13 threads to the inch. Here

$$\epsilon = \frac{D}{F} = \frac{13}{2} = \frac{130 \times 90}{20 \times 90},$$

which is a good train for the purpose. The wheels for forming a series of such trains, calculated for the different numbers of threads to be produced, are known as a set of *change-wheels;* and tables for the use of such wheels are furnished by lathe-manufacturers with all screw-cutting lathes.

239. Frequency of Contact between Teeth. — It is sometimes a matter of interest to know how often any two given teeth will come into contact as the wheels run upon each other. We will take the case of a wheel of A teeth driving one of B teeth, where A is greater than B, and let $\frac{A}{B} = \frac{a}{b}$ when reduced to its lowest terms.

It is evident that the same points of the two pitch circles would be in contact after a revolutions of B, or b revolutions of A. Hence, the smaller the numbers which express the velocity ratio of the two axes, the more frequently will the contact of the same teeth occur.

1. Let it be required to bring the same teeth into contact as *often* as possible.

Since this contact occurs after b revolutions of A, or a revolutions of B, we shall effect our object by making a and b as small as possible; this is, by providing that A and B shall have a large common divisor.

For example, assume that the comparative angular velocity of the two axes is intended to be as nearly as possible as 5 to 2. Now make $A = 80$, $B = 32$; then

$$\frac{A}{B} = \frac{80}{32} = \frac{5}{2} \text{ exactly;}$$

or, the same pair of teeth will come in contact after 5 revolutions of B, or 2 of A.

2. Let it be required to bring the same teeth into contact as *seldom* as possible.

Now change A to 81, and we shall have $\frac{A}{B} = \frac{81}{32} = \frac{5}{2}$ very nearly; or, the angular velocity of A relatively to B wlll be scarcely distinguishable from what it was originally. But the alteration will effect what we require, for now $\frac{a}{b} = \frac{81}{32}$. There will, therefore, be a contact of the same pair of teeth only after 81 revolutions of B, or 32 revolutions of A.

The insertion of a tooth in this manner was an old contrivance of millwrights to prevent the same pair of teeth from meeting too often, and·was supposed to insure greater regularity in the wear of the wheels. The tooth inserted was called a *hunting cog*, because a pair of teeth, after being once in contact, would gradually separate, and then approach each other by one tooth in each revolution, and thus appear to hunt each other as they went round.

Clockmakers, on the contrary, appear to have adopted the opposite principle; though it has probably been partly forced on them, as the velocity ratio of the clock arbors must necessarily be exact.

240. Approximate Numbers for Trains. — If $\frac{L_m}{L_1} = k$, when k is a prime number, or one whose prime factors are too large to be conveniently employed in wheelwork, an approximation may be resorted to. For example, assume $\frac{L_m}{L_1} = k \pm h$. This will introduce an error of $\pm h$ revolutions of the last axis during one of the first, and the nature of the machinery in question can alone determine whether such a variation is permissible.

For example, let $\epsilon = \frac{L_m}{L_1} = 269$, which is a prime number. Take $\epsilon = 269 + 1 = 270$, which can readily be factored into $6 \times 5 \times 9$; and we may employ the train $\frac{D}{F} = \frac{72 \times 60 \times 90}{12 \times 12 \times 10}$. This train will cause an error of one revolution of the last axis for every revolution of the first axis, the altered value of ϵ varying less than two-fifths of one per cent from the correct value.

241. But we may obtain a better approximation than this, without unnecessarily increasing the number of axes in the train; for, determine, in the manner already explained, the least number m of axes that would be necessary if k were decomposable, and the number of teeth that the nature of the machine makes it practicable to give to the pinions, and let F be the product of the pinions so determined; hence

$$\frac{L_m}{L_1} \text{ or } \frac{D}{F} = k = \frac{Fk}{F},$$

supposing the wheels to drive.

Assume

$$\frac{D}{F} = \frac{Fk \pm h}{F},$$

where h must be taken as small as possible, but so as to obtain for $Fk \pm h$ a numerical value decomposable into factors. There will be, in this case, an error of $\pm h$ revolutions in the last axis during F of the first, or an error of $\frac{\pm h}{F}$ during one of the first. If the pinions are to be the drivers, then, in the same manner, assume

$$\frac{L_1}{L_m} = \frac{Dk \pm h}{D};$$

and there will then be an error of $\frac{\pm h}{D}$ revolutions in the first axis during one revolution of the last axis. Let us take, as in the previous example, $\epsilon = 269$. Let $w = 90$, and $p = 10$; then

$$269 = (9)^{n-1},$$

whence we find the least number of axes to be four.

Let us assume that pinions of 10 will be employed; then

$$\epsilon = \frac{D}{F} = 269 = \frac{269000}{10 \times 10 \times 10}.$$

Now add 1 to the numerator, and we have

$$\frac{D}{F} = \frac{269001}{10 \times 10 \times 10} = \frac{81 \times 81 \times 41}{10 \times 10 \times 10}.$$

This will give a good train with an error of only 1 revolution in 269000.

As another example, let it be required to find a train that shall connect the twelve-hour wheel of a clock with a wheel revolving in a lunation (viz., 29 days, 12 hours, 44 minutes nearly), for the purpose of showing the moon's age on a dial.

Reducing the periods to minutes, we have

$$\epsilon = \frac{L_m}{L_1} = \frac{42524}{720},$$

of which the numerator contains a large prime; viz., 10631; but

$$\frac{42524 + 1}{720} = \frac{60 \times 63}{8 \times 8},$$

giving a good train, with an error of one minute in a lunation.

CHAPTER XIII.

AGGREGATE COMBINATIONS.

Differential Pulley.—Differential Screw Feed Motions.—Epicyclic Trains.—Parallel Motions.—Trammel.—Oval Chuck.

242. Aggregate Combinations is the term applied to those assemblages of pieces in mechanism in which the motion of a follower is the resultant of the motions it receives from more than one driver. The number of drivers which impress their motion directly upon the follower is generally two, and cannot exceed three, since each driver determines the motion of at least one point of the follower, and the motion of three points in a body determines its motion.

Such combinations enable us to produce by simple means very rapid or very slow *velocities*, and complex *paths*, which could not well be obtained directly from a single driver. These combinations may be divided into two classes, according as *velocity* or *path* is the principal object to be attained; and we will consider these two classes separately.

AGGREGATE VELOCITIES.

243. By Linkwork.—In Figs. 170 and 171, let AB be a rigid link, and let the point A be given a velocity a, while the point B is given the velocity b. Then it is required to determine the motion of an intermediate point, C, which is affected by the motions of both A and B. These motions are generally perpendicular to AB, or so nearly so that the error in their comparative motions will not generally be practically appreciable.

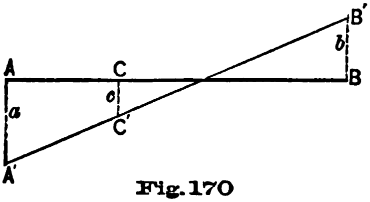


Fig. 170

If we consider the motion of A alone, regarding B as stationary, C will move with a velocity $= a.\frac{BC}{AB}$. If we consider the effect of the motion of B alone, regarding A as stationary, we have the velocity of $C = b.\frac{AC}{AB}$. Considering motion in one direction as positive, and in the opposite direction as negative, we have for the resultant motion of C from both A and B, $c = \frac{a.BC + b.AC}{AB}$, or the *algebraic* sum of the two component velocities.

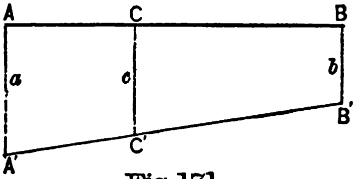


Fig. 171

This result may be represented graphically, as follows: Perpendicularly to AB draw AA' and BB' to represent in length and direction the velocities of A and B respectively. Draw $A'B'$. Then CC' drawn through C perpendicularly to AB will represent in length and direction the resultant velocity of the point C.

Examples of aggregate motion by linkwork are to be seen in the several forms of "link motion" valve gears of reversible steam-engines. In these, motion is given by eccentrics or cranks to points such as *A* and *B* in the figures, and the steam-valve receives its motion from some intermediate point, the distance of which from the ends can be varied. As will be seen from the figures, if *C* is nearer *A* than *B*, for instance, its motion will be derived to a greater extent from *A* than from *B*. If it is midway between these points, it will receive an equal proportion from each.

244. Differential Pulley. — In Weston's differential pulley, illustrated by Fig. 172, the principle of aggregate

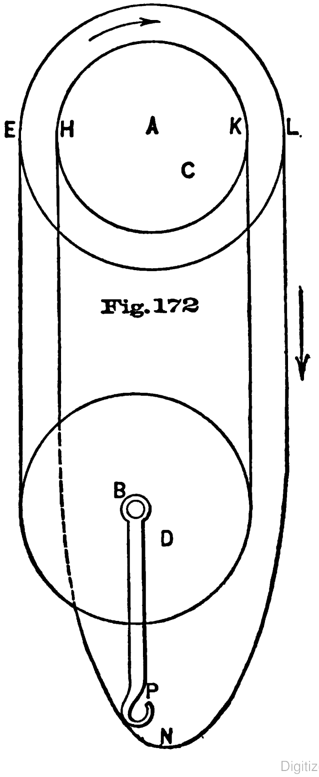


Fig. 172

velocities is made use of for lifting heavy weights by the application of a small amount of force. It consists of a single movable pulley, D, from the axis of which the weight to be lifted is suspended; a fixed pulley, C, having two circumferential grooves, the diameter of one being somewhat less than that of the other; and an endless chain passing around the pulleys, as shown in the figure. The combination is operated by hauling upon the chain LN in the direction indicated by the arrow. The velocity of the pitch circle, EL, is evidently equal to that of the hauling part of the chain. Let l, k, denote the velocities of the pitch circles EL and HK respectively, and b the velocity of BP.

Then, if the point K were stationary, hauling down upon LN would evidently *raise* B with a velocity $= \frac{l}{2}$. But K, being rigidly connected to L, moves *downward* with a velocity such that $\frac{k}{l} = \frac{AK}{AL}$, or $k = l.\frac{AK}{AL}$. Considering E as fixed, this would give to B a downward velocity of $\frac{k}{2}$. Hence the resultant velocity of B upwards will be

$$b = \frac{l}{2} - \frac{k}{2} = l.\frac{AL - AK}{2AL},$$

or the velocity ratio $= \frac{b}{l} = \frac{AL - AK}{2AL}$.

245. Compound Screws. — In Fig. 173 let SS' be a cylinder upon which two screw threads are formed. Let the portion ab have a pitch n, and be fitted in a fixed nut N; and let the portion cd have a pitch m, and be fitted with a nut M which is free to move in the direction SS', but which is prevented from turning. Then, if the bolt be turned in the nuts as indicated, it will move through the nut N, a distance n, during each turn, while at the same time the nut M

will move along *SS′*, a distance *m*, during each turn. Therefore, if the screws wind the same way, *M* will move relatively to the fixed nut *N*, a distance equal to the difference between *n* and *m* for each turn of *SS′*. That is, if *n* is greater than *m*, *M* will move *away from N* the distance $n - m$ for each turn; or if *m* is greater than *n*, *M* will move *towards N* the distance $m - n$. If the screws wind in opposite ways, the motion of *M* relatively to *N* will be $n + m$ for each turn.

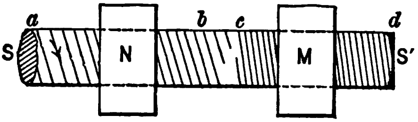


Fig. 173

246. Automatic Drill Feed.—Fig. 174 illustrates a combination for the production of a slow endlong motion of a spindle, together with a rapid rotation such as is required for the spindle of a drill-press. In the figure, *AB* is the spindle to which is fastened the spur wheel *E*. A thread is cut on a portion of *AB*, to which is fitted a nut *N* mounted in the frame of the machine, so that it is free to rotate, but can have no other motion. To *N* is fixed a spur wheel *F*. *E* and *F* gear respectively with a long pinion *H* and a spur wheel *K*, both fixed to a driving-shaft *CD*. Let *c* be the

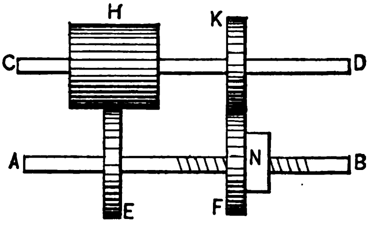


Fig. 174

number of revolutions made by CD, while F and E make f and e revolutions respectively. Also, let E, F, H, and K represent the number of teeth upon the respective wheels. Then, $\frac{e}{c} = \frac{H}{E}$, and $\frac{f}{c} = \frac{K}{F}$. Let p be the pitch of the screw, then c revolutions of CD will cause AB to travel through the distance $(f - e)p = cp\left(\frac{K}{F} - \frac{H}{E}\right)$.

For example, let $p = \frac{1}{4}''$, $\frac{H}{E} = \frac{7}{10}$, and $\frac{K}{F} = \frac{3}{4}$; then, for one turn of CD, the spindle will travel $\frac{1}{4}''(\frac{3}{4} - \frac{7}{10}) = \frac{1}{4}'' \times \frac{1}{20} = \frac{1}{80}''$.

247. An Epicyclic Train is a train of mechanism, the axes of which are carried by a revolving arm. Simple forms of epicyclic trains are illustrated by Figs. 175 and 176. In both figures the train-bearing arm, A, revolves about a fixed centre, B, and carries the train of wheels shown. C, which is considered to be the first wheel of the train, is concentric with B, and may be fixed, or may receive motion from some external source. The wheel E, which is considered to be the last wheel of the train, may be carried by the arm, as in Fig. 175, or be concentric with it, as in Fig. 176. In the latter case it is carried by a separate shaft,

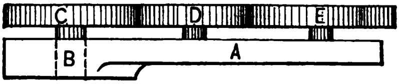


Fig. 175

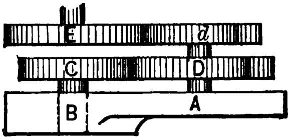


Fig. 176

or turns loosely upon *B*. In either case its actual motion is the resultant of the motion derived from the revolution of the arm *A* and that received from *C* by means of the connecting train. It will be seen that the connection between *C* and *E* may be made by any of the modes of transmitting motion which have been discussed.

Epicyclic trains are used: (1) To produce an aggregate motion of the last wheel by means of simultaneous motions given to the first wheel and the arm. (2) To produce an aggregate motion of the arm by means of simultaneous motions given to the first and last wheels.

248. Velocity Ratio in Epicyclic Trains. — In Fig. 177 let *A* be the train-bearing arm of an epicyclic train turning about *B*. Let *C* be the wheel concentric with *B*, and *E* the axis of a wheel *F* carried by the arm and connected to *C* by a train of mechanism. Suppose that while *A* turns about *B* to some other position *A′*, a point *a*, on wheel

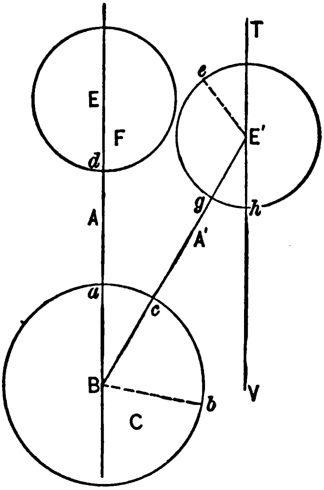


Fig. 177

C, moves to b from any external cause, and that a point d, on wheel F, moves to e by reason of its connection with C. For simplicity, all are supposed to turn in the same direction. Draw $TE'V$ parallel to EB. Then aBb and $hE'e$ are the *absolute* angular motions of C and F respectively, and cBb and $gE'e$ are their angular motions *relatively* to the *arm* A.

$$hE'g = aBc = \text{angular motion of the arm.}$$
$$aBb = aBc + cBb.$$
$$hE'e = hE'g + gE'e = aBc + gE'e.$$

Or,

$$cBb = aBb - aBc; \qquad gE'e = hE'e - aBc.$$

These equations are true for angles of any magnitude, and hence for complete revolutions since the velocity ratio is constant.

Let a, m, and n be the synchronal *absolute* rotations of the arm, of the first wheel C, and of the last wheel F respectively. Let ϵ be the value of the train between C and F, that is the quotient which has been represented by $\frac{L_m}{L_1} = \frac{D}{F}$ in Chap. XII. Then the rotations of the first wheel relatively to the arm $= m - a$, and the rotations of the last wheel relatively to the arm $= n - a$. Therefore $\epsilon = \frac{n - a}{m - a}$, which is the general equation for epicyclic trains.

From this we derive

$$a = \frac{m\epsilon - n}{\epsilon - 1}, \quad m = a + \frac{n - a}{\epsilon}, \quad n = a + \epsilon(m - a).$$

If the first wheel is fixed, $m = 0$.

$$\therefore \quad \epsilon = \frac{a - n}{a}, \quad a = \frac{n}{1 - \epsilon}, \quad n = (1 - \epsilon)a.$$

If the last wheel is fixed, $n = 0$.

$$\therefore \quad \epsilon = \frac{a}{a - m}, \qquad a = \frac{m\epsilon}{\epsilon - 1}, \qquad m = \left(1 - \frac{1}{\epsilon}\right)a.$$

In all of the above formulæ, the arm, first wheel, and last wheel are assumed to rotate in the same direction; but if the direction of rotation of any one is changed, the sign of a, m, or n should be changed accordingly. In applying the formulæ, we first assume that the rotations take place in the same direction, and then, one direction for the arm being taken as positive, the $+$ or $-$ sign of m and n will show whether they are rotating in the same direction or the reverse.

If the connecting train is such that the first and last wheels would rotate in the same direction, supposing the arm to be fixed, the sign of ϵ is *plus*, but if they would rotate in opposite directions, it is to be taken as *minus*. For example, if the connection is by spur gearing, and there are an *odd* number of axes, ϵ is *positive;* but if the number of axes is *even*, ϵ is *negative*.

249. Ferguson's Paradox, illustrated by Fig. 178, will serve as a simple example for the application of these formulæ. The wheel C has 20 teeth, and is fixed to the shaft B, about which the arm A rotates. This arm carries the axis of the wheel D, which gears with C and with three wheels E, F, and G, which turn loosely on the shaft H also carried by the arm. E has 19 teeth, F 20, G 21, and D any number. Since there are three axes, ϵ is $+$, and has the three values, $\frac{C}{E} = \frac{20}{19}$, $\frac{C}{F} = \frac{20}{20}$, and $\frac{C}{G} = \frac{20}{21}$. C is fixed; therefore, $m = 0$, and $n = (1 - \epsilon)a$.

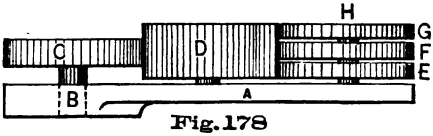


Fig. 178

In the three cases we have

$$(E)\qquad n = \left(1 - \frac{20}{19}\right)a = -\frac{1}{19}\,a;$$

$$(F)\qquad n = \left(1 - \frac{20}{20}\right)a = 0;$$

$$(G)\qquad n = \left(1 - \frac{20}{21}\right)a = +\frac{1}{21}\,a.$$

That is, when the arm revolves the wheel *F* will have no absolute rotation, while, for each revolution of the arm, *E* will make $\frac{1}{19}$ of a turn in the opposite direction, and *G* will make $\frac{1}{21}$ of a turn in the same direction.

250. Watt's Crank Substitute, otherwise known as the *Sun and Planet Motion*, belongs to the general class of epicyclic trains. In Fig. 179, *AB* is one end of the main beam of an engine, *C* is a spur wheel fastened to the main shaft, and *E* is a spur wheel fastened to the connecting-rod *BD*, and gearing with *C*. *E* is held in gear with *C* by means of a connecting link *OD*, or by a circular groove concentric with *C* in which a pin at *D* slides. As *E* is raised and lowered by the motion of the beam, and forced to revolve about *C*, since it cannot rotate its own axis, it causes *C* to rotate. *E* has a vibratory motion due to the varying angle of the connecting-rod, but as this is periodic, it may be neglected for complete revolutions.

Considering the combination as an epicyclic train, *OD* will be the train-bearing arm, *C* the first wheel, and *E* the last wheel. The latter has no absolute rotation; hence, applying the general formula, and letting $n = 0$, we have $m = a\left(1 - \frac{1}{\epsilon}\right)$. Also, since there are but two axes, ϵ is negative.

$$\text{Let } C = E,\ \text{then } \epsilon = -1,\ m = a\left(1 - \frac{1}{-1}\right) = \epsilon a.$$

Or, for one revolution of the train arm *OD* corresponding to an up-and-down stroke of the piston, *C* makes two revolutions. Thus by this arrangement the shaft rotates twice as fast as it would with the ordinary crank connection. If *C* has twice as many teeth as *E*, $\epsilon = -2$, and $m = a\left(1 - \frac{1}{-2}\right) = \frac{3}{2}a$, or *C* revolves three times while *OD* revolves twice. If *E* has twice as many teeth as *C*, $\epsilon = -\frac{1}{2}$, $m = a(1 + 2) = 3a$, or *C* revolves three times for one revolution of *OD*.

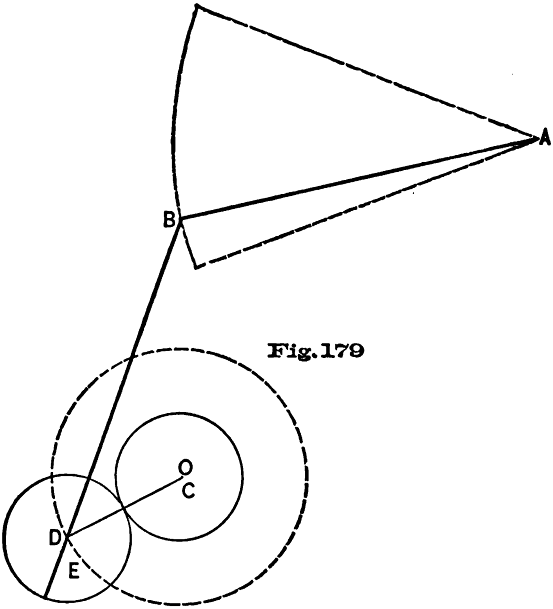


Fig. 179

251. Epicyclic trains are used in some forms of rope-making machinery. In order that a rope shall not untwist, it is necessary that the separate strands shall either be laid together without any twist, as in wire rope, or that they shall have a slight twist in the opposite direction to the apparent twist of the rope. In Fig. 180, let *B* be the bobbins from which the wire or strands are unwound as the rope is formed. These bobbins are carried by wheels *D*, which are connected to a centre wheel *A* by intermediate wheels *C*. The axes of all the wheels excepting *A* are carried by a frame which turns about the axis of *A*. If the bobbins were fixed in this frame, as the frame revolved, each strand would be twisted as it was unwound, but if we arrange it so that the axes of the bobbins shall always lie in the same direction, there will be no twist. This is accomplished by fixing the axes of the bobbins to the wheels *D*, fixing the wheel *A*, and making $D = A$. We then have an epicyclic train in which $m = 0$, and $\epsilon = \frac{-a}{n-a} = \frac{A}{C} \times \frac{C}{D} = 1$, $\therefore\ n - a = -a$, and $n = 0$, or the wheels *D* have no absolute rotation, and consequently there is no twist given to the strands. By giving *D* a few more teeth than *A*, the strands will be given a slight twist in the opposite direction to the twist of the rope.

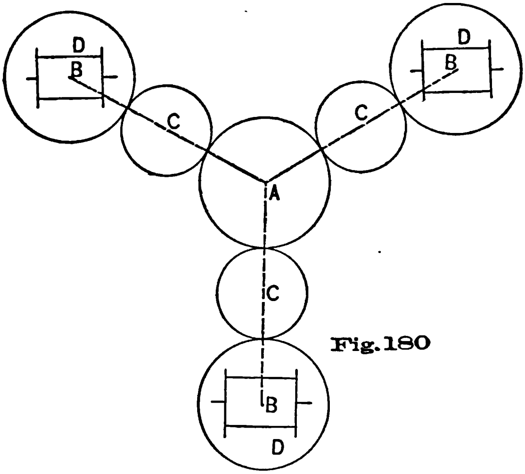


Fig. 180

252. Epicyclic trains may be used to transmit velocity ratios which could not be conveyed by direct trains except by using a large number of axes or inconveniently large wheels. The necessity for such ratios rarely arises except in astronomical machinery, and for explanations of such applications the student is referred to Willis' "Principles of Mechanism," and the works there referred to.

AGGREGATE PATHS.

253. Parallel Motions. — The most important application of aggregate combinations in which the *path* is the immediate object sought, is to give motion to a piece such that a point in it shall move in a straight line. Such combinations are commonly called "parallel motions," although "straight-line motion" would be a more correct and descriptive name.

Some of these combinations give an *exact* straight-line motion, but in most of them the motion is only approximate. We have seen an example of exact straight-line motion in the case of a point on the circumference of a circle rolling within another circle of twice its diameter, being in fact a special case of the hypocycloid. By means of accurately cut gears, this could, of course, be applied to machinery.

In the parallel motions in general use, the straight-line path is produced by combinations of links, and such combinations will be now considered.

254. Peaucellier's Exact Straight-Line Motion. — In Fig. 181 is shown the general arrangement of Peaucellier's *exact* straight-line motion. It consists of seven movable links connected as shown. Two long links AD, AE, oscillate about a fixed centre A, and are jointed at the ends D and E to opposite angles of a rhombus, $CDPE$, composed of four shorter links. At C is connected a link BC, oscillating about a fixed centre B, so located that $AB = BC$.

Then the point *P* will describe a straight line perpendicular to *AB*.

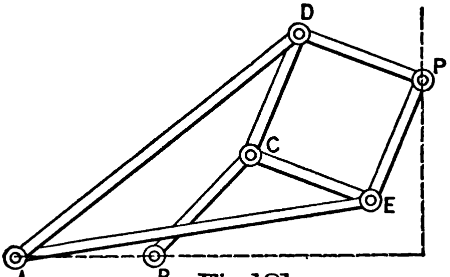


Fig. 181

From the symmetrical construction of the combination it is evident that the points *A*, *C*, and *P* must always lie in one straight line. Let the combination be moved, Fig. 182,

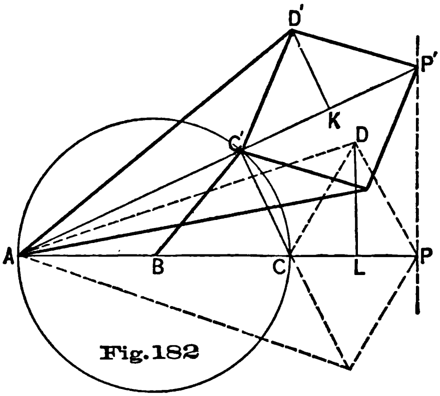


Fig. 182

from the central position shown dotted, to some other position, such as that shown in full lines, the point *P* occupying the position *P'*. Draw *AP*, *AP'*, and *CC'*; also *DL* perpendicular to AP, and $D'K$ perpendicular to AP'. From the construction, $P'K = KC'$, and $PL = LC$. Then,

$$\overline{AD'}^2 = \overline{AK}^2 + \overline{KD'}^2 = \overline{AK}^2 + (\overline{D'C'}^2 - \overline{KC'}^2);$$

$$\begin{aligned}\overline{AD'}^2 - \overline{D'C'}^2 &= \overline{AK}^2 - \overline{KC'}^2 \\ &= (AK - KC')(AK + KC') = AC' \times AP'.\end{aligned}$$

Similarly,

$$\overline{AD}^2 = \overline{AL}^2 + \overline{DL}^2 = \overline{AL}^2 + (\overline{DC}^2 - \overline{LC}^2);$$

$$\begin{aligned}\overline{AD}^2 - \overline{DC}^2 &= \overline{AL}^2 - \overline{LC}^2 \\ &= (AL - LC)(AL + LC) = AC \times AP.\end{aligned}$$

$$\therefore \quad AC \times AP = AC' \times AP';$$

or

$$\frac{AP}{AP'} = \frac{AC'}{AC}.$$

AC is a diameter of the circle $AC'C$; hence $CC'A$ is a right angle, and $P'P$ is perpendicular to ABP. And P' having been assumed as any position of P, it follows that the above relation is true for all positions, or P moves in a straight line perpendicular to AB.

255. In applying this motion to engines, the point P is connected to the end of the piston-rod, and thus takes the place of the usual cross-head and guides. It is to be particularly noted, that, as stated above, the arm BC is equal in length to the distance AB. If this is not so, instead of a straight line, circular arcs will be described by P. If the ratio $\frac{AB}{BC}$ is *less* than *one*, the arc described will be *concave* towards A; if the ratio is *greater* than *one*, the arc described will be *convex* towards A; and if the ratio is *equal* to *one*, the circular arc becomes a *straight line*.

There are other *exact* parallel motions* formed by combinations of linkwork, most of which are derived from the Peaucellier cell; but they are of so little practical importance that they will not be discussed in these pages.

256. Watt's Approximate Straight-Line Motion.— The most widely used of the approximate straight-line motions is that invented by James Watt. It is shown in its simplest form in Fig. 183. *AC* and *BD* are two arms turning about fixed centres *A* and *B*, and connected by a link *CD*. When in the mid position the arms are parallel, and *CD* is perpendicular to them. If the arms be made to oscillate, a point in *CD*, such as *P*, will describe a figure similar to that shown. But we can so arrange the proportions of the links, and the position of *P*, that for a limited motion it will not deviate much from a straight line.

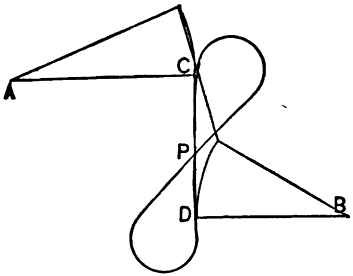


Fig. 183

257. Let the arms *AC* and *BD* be turned to some other positions, as *Ac* and *Bd* in Fig. 184. Then the link *CD* will be moved to *cd*. The end *C* has been moved to the right, and the end *D* to the left, so there will be some point *P*, of *cd*, which will lie in the continuation of the line *CD*. Let

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* For description of parallel motions referred to, see A. B. Kempe's "How to Draw a Straight Line."

$AC = R$, $BD = r$, $CAc = \theta$, $DBb = \phi$, $CD = l$, and $cP = x$. Drawing ce and dg parallel to AC, we have

$$\frac{cP}{dP} = \frac{x}{l-x} = \frac{ce}{dg} = \frac{R(1-\cos\theta)}{r\,(1-\cos\phi)}$$

$$= \frac{2R\sin^2\frac{\theta}{2}}{2r\sin^2\frac{\phi}{2}} = \frac{r}{R}\cdot\frac{R^2\sin^2\frac{\theta}{2}}{r^2\sin^2\frac{\phi}{2}}.$$

In practice, θ does not exceed about 20°, the inclination of the link cd is small, and $R\theta$ is very nearly equal to $r\phi$. As these angles are small, we may assume $R\sin\frac{\theta}{2} = r\sin\frac{\phi}{2}$,

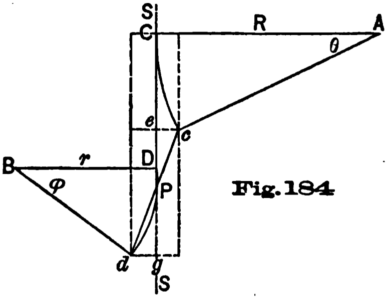

Fig. 184

hence $\frac{x}{l-x} = \frac{r}{R}$, or the segments of the link are inversely proportional to the lengths of the nearest arms, which is the usual practical rule.

258. Amount of Deviation. — The deviation of the point P from the line SS can be calculated, but will not generally exceed about $\frac{1}{10}$ inch. This may be greatly reduced by the arrangement shown by Fig. 185, which should always be used. In the mid position the arms are perpendicular to the line SS in which the point P should lie, and which in an engine should coincide with the centre line of the cylinder or pump. This line should bisect the distances *Ce* and *Df* which are the versed sines of the maximum values of the angles θ and ϕ. The ends *C* and *D* of the link will then evidently deviate equal amounts on each side of *SS*. Drawing *d′h* and *dg* perpendicular to *SS*, and *Cn* parallel to *SS*, we have three equal triangles, *c′d′h*, *CDn*, and *cdg*. Therefore, $c'p' = CP = cp$, or the mid and extreme positions of the guided point *P* are exactly on *SS*.

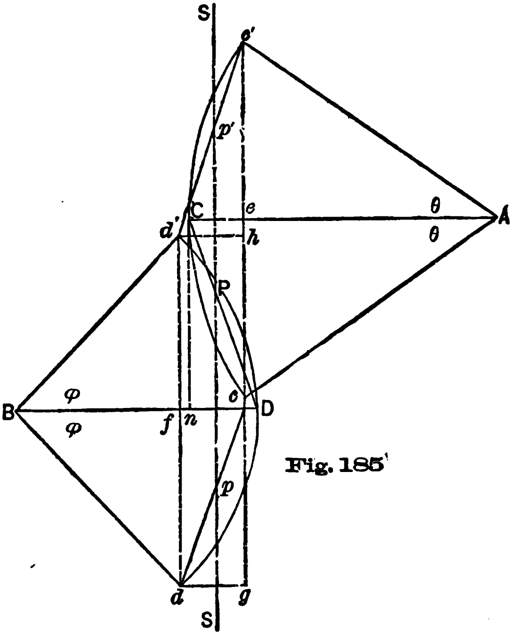

Fig. 185

The greatest deviation of the guided point from *SS* occurs when *CD* is parallel to *SS*, and is best determined in any case by drawing the combination to a large scale, and finding the parallel position by trial.

259. Problem.—In Fig. 186, let *CA* be an arm as before, *cA* its extreme position, and *SS* the line of stroke bisecting *Ce*. Join *Cc*, and draw *AN* perpendicular to it. *N* bisects *Cc*, since the latter is the chord of the angle *CAc*, and hence is on the line *SS*. Also $MN = \frac{1}{2}\, ec$, or, since *ec* may be taken as $\frac{1}{2}$ the stroke, $MN = \frac{1}{4}$ the stroke.

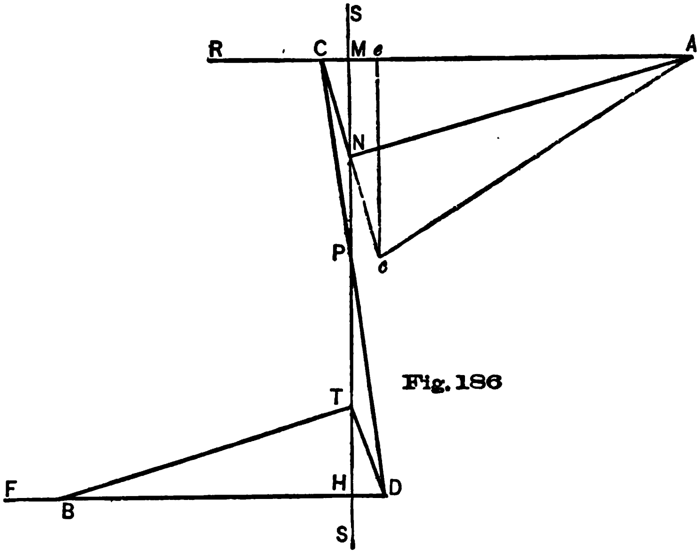

Fig. 186

Therefore, if we have given the length of stroke and direction, *SS*, the centre of one arm *A*, and mid position of the guided point *P*, we can construct the remainder of the motion as follows: Draw *AR* perpendicular to *SS*, lay off $MN = \frac{1}{4}$ stroke, draw *AN*, and perpendicular to the latter draw *NC*. Where this line intersects *AR* at *C*, will be the end of the arm *AC*. *CP* will be the direction of the link in mid position. If we assume, or have located, the point *H* where the mid position of the second arm cuts *SS*, draw an indefinite straight line, *FH*, through this point perpendicular to *SS*. The point *D*, where *CP* produced cuts *FH*, is the extremity of the second arm. Then, since *HD* must be $\frac{1}{2}$ the versed sine of the arc through which *D* moves in either direction, we can find the centre *B* by laying off $HT = \frac{1}{4}$ stroke, and drawing *TB* perpendicular to *TD*.

260. Practical Form of Watt's Motion. — We have thus found the proper proportions for the simplest form of the motion; but, as usually constructed, the motion is of the form shown in Fig. 187. *AE* is one arm of the main beam

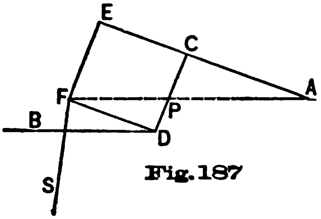

Fig. 187

of an engine, and turns about the centre *A*. *EF* is the *main link*, connecting *AE* with the piston-rod *FS*. *CD* is the *back-link* equal and parallel to *EF*. *FD* is the *parallel-bar* equal and parallel to *EC*. *BD* is the *radius bar*, or *bridle*. The point *P*, in *CD*, is the guided point whose motion we have discussed. If we draw *AP*, and produce it until it cuts *EF* in *F*, the latter point will have a motion similar to *P*. This will be clear when we consider that in all positions *EF* is parallel to *CP*; then, since *AE* and *AC* are fixed lengths, we have for any position two similar triangles *ACP* and *AEF*; hence $\frac{AP}{AF} = \frac{AC}{AE} =$ constant. So that, if *P* describes a straight line, *F* will also move in a straight line parallel to the path of *P*.

261. Scott Russell's Motion. — A combination due to Mr. Scott Russell, similar to that of Fig. 120, is usually classed as an exact straight-line motion. In that figure, if the point Q be compelled to move in straight guides along AL, the point V will move in a straight path AV, the arm AP oscillating instead of performing complete revolutions. This would scarcely seem to be entitled to the term "exact motion," since it depends upon the accuracy of the guides at Q, the necessity of which it is the object of straight-line motions to avoid.

262. Grasshopper Motion. — A form of the above motion in which the guides are replaced by a comparatively long radius-rod perpendicular to AL in mid position, and connected to Q, is approximate, and is known as the "Grasshopper Motion."

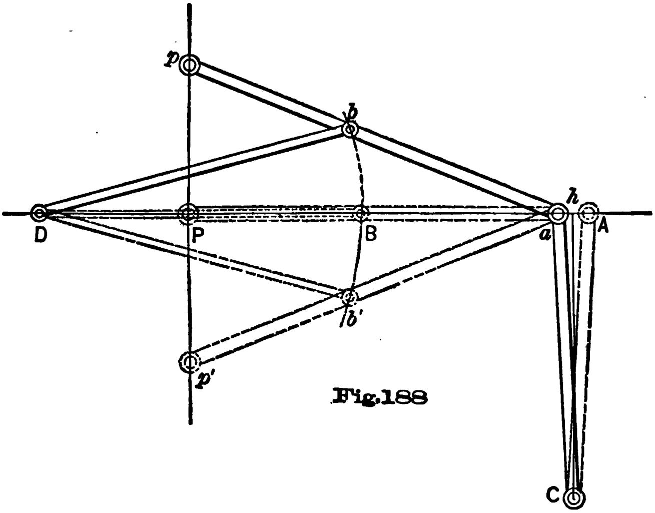

Fig. 188

In Fig. 188, let p, P, and p' be the extreme and middle positions of the guided point, lying in one straight line. Draw the straight line DPB, perpendicular to pPp'; and lay off $pa = p'a = PA =$ the proposed length of the guiding bar, so as to find the extreme positions *A* and *a* of its farther end. This end is to be guided by a lever centred at *C;* that lever being so long as to make the point *A* describe a very flat circular arc, deviating very little from a straight line.

Choose a convenient point *b* for the attachment of the bridle to the bar *AB*, and lay off $pb = p'b' = PB$, so as to find the extreme and middle positions of that point. Next find the centre *D* of a circular arc passing through *b*, *B*, and *b'*; then *D* will be the axis of motion of the bridle *Db*. The error of this parallel motion is less, as *b* is nearer the middle of *pa*.

263. Robert's Approximate Straight-Line Motion. — Fig. 189 illustrates Robert's parallel motion. Two equal arms *AC* and *BD* are jointed to fixed centres at one end, connected at the other end to the ends of the base of a rigid isosceles triangle *CPD*. In this triangle, $CP = DP = AC = BD$, and $CD = \frac{1}{2}AB$. It is evident that in the mid position shown, the point *P* is in the straight line *AB;* also, that it will lie in this line when *PD* coincides with *BD* at one

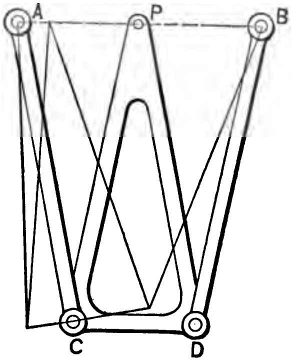

Fig. 189

end of the stroke, and when PC coincides with AC at the other end of the stroke. Between these positions, however, P deviates slightly from AB.

264. Tchebicheff's Approximate Straight-Line Motion. — Another close approximation to a straight-line motion is that due to Prof. Tchebicheff of St. Petersburg, and illustrated by Fig. 190. The arms are of the following proportions: Let $AB = 4$, then $AC = BD = 5$, and $CD = 2$.

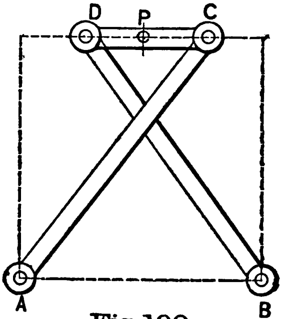

Fig. 190

The path of the guided point P, midway between C and D, will then closely approximate to a straight line parallel to AB. It may be easily proved that the distance of P from AB is the same at the ends of the stroke, where P is in the perpendiculars to AB through A and B, and in the mid position being that shown in the figure. In intermediate positions P deviates slightly from a straight line. Both this and the preceding motion give a closer approximation than can be obtained by Watt's motion.

265. A Trammel is a device for drawing ellipses. It consists (Fig. 191) of a bar, PCD, carrying a pencil at P, and fitted with pins, or pieces mounted on pins, which slide in grooves, as shown in the figure. The grooves are usually at right angles with each other, and the cross-shaped piece in which they are formed is fastened in place on the paper. Let $PD = a =$ the semi-major axis of the ellipse to be drawn, $PC = b =$ the semi-minor axis, $PM = x$, and $PN = y$. Then we have

$$\frac{PM}{PD} = \frac{x}{a} = \sin PDM = \sin\phi;$$

$$\frac{PN}{PC} = \frac{y}{b} = \cos CPN = \cos\phi.$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2\phi + \cos^2\phi = 1,$$

which is the equation of an ellipse. By varying the lengths PC and PD, ellipses of different sizes and eccentricities can be drawn.

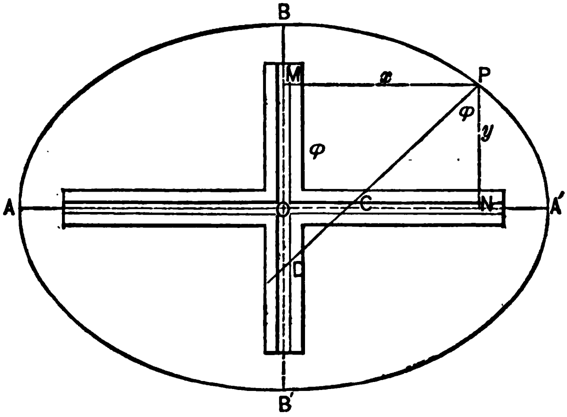

Fig. 191

266. Oval Chuck.—If in Fig. 192 we keep the bar CPD stationary, and turn the grooved piece and paper, an ellipse will be described upon the paper by the point P as before. This fact is taken advantage of in the so-called "oval" chuck for turning ellipses, and of which Fig. 192 illustrates the principle. In this figure P is the cutting tool, C the centre of the mandrel of the lathe, and D the centre of a circular piece which is fixed to the headstock of the lathe. One part of the chuck is fixed to the mandrel, and has cut in it a diametral slot represented by aCb. A second

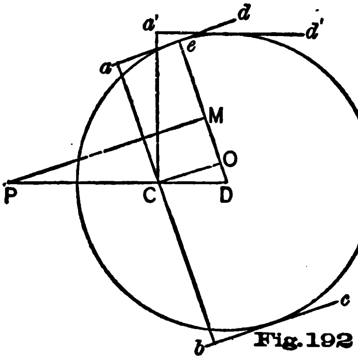

Fig. 192

part of the chuck, being that which carries the piece to be turned, has two lugs which project through the slot aCb and form part of two straight pieces, represented by ad and bc, which slide on the circular piece previously referred to. The result is, that, as the mandrel revolves, the piece being turned, or the work, receives a combination of this motion of rotation and a reciprocating motion in the slot, by which the distance of the centre of the work from the tool is varied in the manner necessary to form an ellipse. Draw De parallel to Ca, and CO perpendicular to Ca. Then when the work has been turned about C through the angle $a'Ca$, it has also been moved through C the distance OD. We now see that the triangle COD of Fig. 192 corresponds to COD

of Fig. 191, and drawing PM perpendicular to De, we have, as before,

$$\frac{PM}{PD} = \sin PDM = \sin \phi ;$$

$$\frac{OM}{PC} = \cos PDM = \cos \phi ;$$

or O is for the instant the centre of the ellipse. Evidently since P, C, and D are fixed, the position of this centre is constantly changing, lying always at the junction of a perpendicular to aCb through C, and a parallel to aCb through D.

PROBLEMS.

1. An engine makes 600 strokes per minute. Fly-wheel is on the crank shaft. Find the linear and angular velocity of a point in the fly-wheel 3 feet from the centre of the shaft.

Ans. $a = 1884.96$; $V = 5654.88$ feet per minute.

2. The speed of the periphery of a wheel 8 feet in diameter is 4,000 feet per minute. Find the linear velocity of a point $3\frac{1}{2}$ feet from the centre.

3. A point in a fly-wheel, 4 feet from the centre of the wheel, moves through 2,500 feet per minute. The stroke of the engine being 2 feet, find the mean piston speed.

Ans. $V = 397.89$ feet per minute.

4. A locomotive moving at the rate of 35 miles per hour, has driving wheels 63 inches in diameter and cylinders 24 inches stroke. Find the linear and angular velocities of the crank-pins relatively to the frame of the engine.

5. Two shafts are centred 2 feet apart. Find the diameters of wheels to work by rolling contact, so that the driving shaft will make 5 revolutions while the following shaft makes 7 revolutions.

Ans. Driver, 28 inches; follower, 20 inches.

6. The distance between the centres of two shafts = 54 inches. The driving-shaft makes 80 revolutions per minute. The follower is to make 100 revolutions per minute. Find the diameter of wheels for rolling contact.

7. A shaft making 120 revolutions per minute is to drive by spur gearing a second shaft 28 inches from it at a speed of 300 revolutions per minute. Find diameters of pitch circles.

8. Velocity ratio to be transmitted $= \frac{3}{4} = \frac{a'}{a}$. Diameter of the driver is 15 inches. Find the diameter of the follower, and the distance between parallel axes. (Direct contact.)

Ans. Diameter, 20 inches; distance, 17½ inches.

9. A wheel 32 inches in diameter is fixed on a shaft making 325 revolutions in 5 minutes. This wheel and shaft are to drive a second wheel by rolling contact, so that the latter will make 52 revolutions per minute. Find the size of the second wheel, and the distance between the centres of the wheels.

10. Given two intersecting axes at right angles, velocity ratio $\frac{a'}{a} = \frac{4}{3}$. Show how to find the pitch cones graphically.

11. The angle between two intersecting axes is 75°. Show how to find graphically the sizes and positions of conical frusta which will transmit a velocity ratio $\frac{a'}{a} = \frac{65}{80}$.

12. P = circular pitch, N = number of teeth.
D = pitch diameter, M = diametral pitch.

(1). Given $P = 2\frac{1}{2}$ inches, $N = 40$. Find D.
(2). Given $P = 1\frac{1}{2}$ inches, $N = 75$. Find D.
(3). Given $P = \frac{3}{4}$ inch, $D = 12$ inches. Find N.
(4). Given $D = 24$ inches, $N = 50$. Find P.
(5). Given 8 pitch wheel, $N = 40$. Find D.
(6). Given 3 pitch wheel, $N = 60$. Find D.
(7). Given 4 pitch wheel, $D = 20$ inches. Find N.
(8). Given 2 pitch wheel, $D = 35$ inches. Find N.
(9). Given $D = 15$ inches, $N = 75$. Find M.
(10). Given $D = 27$ inches, $N = 81$. Find M.

13. Two axes 27 inches apart are to be connected by two 2 pitch wheels. Velocity ratio $\frac{7}{5}$. Find diameters of pitch circles and numbers of teeth.

Ans. Numbers of teeth 63 and 45.

14. Construct a cam on a base circle of 3 inches diameter, to revolve once per minute, and give to a bar, whose line of motion passes through the centre of motion of the cam, a stroke of 2 inches. The bar rises during 25 seconds with a uniform velocity; remains at rest 20 seconds; and descends during the remainder of the revolution with a uniformly accelerated velocity.

15. Draw a cam, which, by oscillating through an angle of 60°, shall give a uniform ascending and descending motion to a bar whose line of motion passes 4 inches to right of the centre of the cam. Stroke of the bar, 3 inches.

16. Design a cam on a base circle of 3 inches diameter, to raise a point whose line of motion passes one inch to the right of the centre of motion of the cam, by a uniform step-by-step motion, during $\frac{3}{4}$ of a revolution of the cam, and allow it to descend with uniform velocity during the remaining $\frac{1}{4}$ of the revolution.

17. In Fig. 107 given $AP = 2$ feet, $AB = 1$ foot, $BC = 3\frac{1}{2}$ feet, $CQ = 6$ feet. AB is vertical, and aQ is horizontal. P revolves in the direction of arrow, making one revolution per minute.

(1). Find length of stroke of Q
(2). Find time of forward stroke in seconds
(3). Find time of backward stroke in seconds
} By computation.

(4). Find position of P when Q is at the middle of forward stroke
(5). Find position of P when Q is at the middle of backward stroke
} Graphically.

18. Having a crank 2 feet long, and a connecting-rod 8 feet long, find the angle of the crank with line of centres when the piston is at the middle of its stroke.

Ans. $\pm 82° 49' 9''$.

19. Having a crank 1 foot long, and a connecting-rod 5 feet long, revolutions per minute 120. Find piston velocity in feet per minute, when the crank makes an angle of 45° with the line of centres.

Ans. 609.22 feet per minute.

20. Having an engine of 5 feet stroke and a 10-foot connecting-rod, find distance of the piston from the end of stroke when the crank has made $\frac{1}{4}$ of a revolution.

Ans. 2 feet 2.19 inches.

21. Having an engine of 6 feet stroke, connecting-rod $10\frac{1}{2}$ feet long, find what angles the crank makes with line of centres when the velocity of the piston equals that of the crank.

Ans. $\text{Sin}^{-1}.\ \frac{7}{8}$.

22. Having a beam engine of 10 feet stroke, 13 feet between the centres of beam and cylinder. Find the best length for the beam arm.

23. Show graphically how to construct a quick return motion by jointed links, such that $\frac{\text{period of advance}}{\text{period of return}} = \frac{3}{2}$.

24. Design a quick return motion, such that the period of return $= \frac{4}{7}$ of the period of advance.

25. Design a cam on a base circle of 2 inches diameter, to give to a point whose line of motion passes ½ inch to the right of the centre of motion of the cam, the same motion as piston in problem 20.

26. Connect two parallel shafts by a crossed belt, so that $\frac{a'}{a} = \frac{1}{4}$, and find the length of the belt by exact calculation.

27. Two shafts are to be connected by an open belt, distance between axes = 10 feet and $\frac{a'}{a} = \frac{2}{3}$. Find diameters of pulleys and the length of the belt.

28. A pulley (A) on a driving-shaft drives pulley (B) by a crossed belt. A spur gear (C) on shaft with (B) drives pinion (D). Pulley (E), on the shaft with (D), drives pulley (F) by an open belt.

Given $A = 20$ inches diameter, 40 revolutions per minute.
Given $B = 15$ inches diameter.
Given $C = 90$ teeth, $D = 15$ teeth.
Given $E = 30$ inches diameter, $F = 10$ inches diameter.

Find number of revolutions per minute of F, and direction of rotation relatively to A.

29. An engine of 3 feet stroke, piston speed of 360 feet per minute, has a main driving pulley 8 feet in diameter, from which is driven a pulley 4 feet in diameter. A pump having a plunger displacement of 2 gallons, is to be driven from the shaft carrying the 4-feet pulley, and is to pump 5,000 gallons per hour. Find arrangement of the connecting train of mechanism, by belts or gearing.

30. A lathe has a set of change wheels whose pitch diameters are 2 inches, 3 inches, 5 inches, 6 inches, 7½ inches, and 9 inches respectively. Leading screw has 4 threads to the inch, and is right-handed. Distance between the centres of spindle and leading screw is 16 inches. Select and arrange wheels to cut a left-handed screw of 6 threads to the inch.

31. A lathe has 4 threads per inch on a right-handed leading screw. Find the sizes of least number of change wheels to cut right-handed threads of 5, 6, 8, 9, and 10 to the inch. Smallest wheel to have 20 teeth. Arrange table for change wheels for the various cuts.

32. Find trains for an 8-day clock, 16 turns of weight cord on barrel. The escape wheel has 30 teeth.

Number of teeth on wheels not to exceed 96
Number of teeth on pinions not less than 8

Required hour, minute, and seconds hands.

33. Find the trains for a 32-day clock, the barrel to carry 24 coils of the weight cord; pinions to have not less than 8, and the wheels not over 108 teeth; swing wheel (escape wheel) to have 60 teeth, and the pendulum to make 120 vibrations per minute. Required hour, minute, and seconds hands.

34. Find trains for a 12-day clock; 18 turns of weight cord on barrel. Escape wheel revolves twice per minute. Pendulum makes 120 beats per minute. Least number of teeth for pinions = 9. Greatest number for wheels = 108. Required hour, minute, and seconds hands.

35. Find trains for an 8-day clock. Pendulum makes 150 vibrations per minute. Swing wheel has 25 teeth. Dead beat escapement. Least number of teeth for pinions = 10. Greatest number for wheels = 108. 12 coils of weight cord on barrel.

36. Find numbers of teeth for a train to give approximately $E =$ 194 with an error of less than 1; maximum number of teeth for wheel = 90; minimum number for pinion = 12.

37. Design a drill press (Fig. 174), pitch of screw to be $\frac{3}{8}$ inch; drill to make 60 revolutions per minute; driving axis to make 40 revolutions per minute. Drill to descend $\frac{1}{64}$ inch per revolution.

38. In Fig. 176, C has 121 teeth, and is fixed, D has 120 teeth, d has 119 teeth, $E = 120$ teeth. Find how many revolutions of arm A will cause E to revolve once.

39. In Fig. 176, C is a fixed wheel, and has 20 teeth, $D = 36$ teeth, $d = 24$ teeth, $E = 32$ teeth. Find velocity ratio $= \frac{E}{A}$.

40. In Fig. 179, C has 30 teeth, and E has 40 teeth. Required velocity ratio $\frac{C}{OD}$.

41. In Fig. 179. If C makes 3 turns while OD makes 5 turns, find number of teeth for E and C.

42. In Fig. 182. Let the arm BC' be suppressed, and let P' be guided in a circle drawn on AP as a diameter. Prove that C' will move in a straight line perpendicular to AP.

43. Find dimensions of a trammel to describe an ellipse of which the major axis is 2¼ times as long as the minor axis.

INDEX.

PAGE

Action, angle and arc of 86
Action, line of 26
Action of bevel and spur wheels, relative 156
Addendum 85
Addendum circle 85
Aggregate combinations 263
Aggregate velocities by linkwork . . 263
Analogy between cones and hyperboloids 42
Anchor escapement 188
Angle of action, approach and recess, 86
Angular velocity 4
Annular wheels 47, 101, 138
Approach, angle and arc of 86
Approximate forms of teeth 113
Approximate numbers for trains . . 260
Approximate straight-line motion, Roberts' 284
Approximate straight-line motion, Tchebicheff's 285
Approximate straight-line motion, Watt's 278
Arc of action, approach and recess . 86
Arc of action and pitch, relation between 95
Arcs, rectification of circular . . . 74
Axes, least number of 250
Automatic drill-feed 267

Bands 15
Backlash 86
Bar and cam, slotted 183
Bar and pin, slotted 182
Bell-crank 209

PAGE

Belts 228
Belts, length of 236
Belts, shifting 231
Belts, twisted 232
Bevel gearing 47, 151
Bevel and spur wheels, relative action of 156
Bevel wheels 151
Bevel wheels, skew 157
Boehm's coupling 208

Cams 165
Cam and slotted bar 183
Cam curve, construction of 166
Cam for complete revolutions . . . 169
Catch, frictional 223
Centre, instantaneous 18
Chuck, oval 286
Circle, addendum 85
  " of the gorge 36
  " pitch 45
  " size of describing 94
Circular arcs, rectification of . . . 74
  " pitch 80
Clearance 86
Click and ratchet 219
  " double-acting 222
  " reversible 220
  " silent 221
Clock, eight-day 256
  " month 257
  " trains, examples of 251
Clockwork 245
Cog, hunting 260
Combination, elementary 7

PAGE
Component motions . . . 11
Composition of motions . . . 11
Condition of constant velocity ratio . 24
" of rolling contact . . . 24
Cones and hyperboloids, analogy between . . . 42
Cones, rolling . . . 29
Conjugate curves . . . 73
Connectors, wrapping . . . 15, 227
Contact between teeth, frequency of . 259
" condition of rolling . . . 24
" motions, directional relation in . . . 24
" motions, velocity ratio in . . 21
" rolling . . . 14, 27
" sliding . . . 14, 66
Coupling, Boehm's . . . 208
" or universal joint, Hooke's, 214
" Oldham's . . . 185
Crank . . . 193
Cranks, bell . . . 209
Crank and pin, slotted . . . 179
Crank substitute, Watt's . . . 272
Crown-wheel escapement . . . 187
Curves, conjugate . . . 73
Curvilinear motion . . . 2
Cycle . . . 6
Cycloid, construction of . . . 79
Cylinders, rolling . . . 27

Dead-beat escapement . . . 190
Derived tooth outline, pin gearing . 133
Describing circle, size of . . . 94
Diameter of pin, limiting . . . 136
" pins of sensible . . . 133
" table of pitch . . . 81
Diametral pitch . . . 82
Differential pulley, Weston's . . . 265
Directional relation . . . 8
Directional relation in contact motions . . . 24
Directional relation in linkwork . . 20
Directional relation in trains . . . 243
Directional relation in wrapping connectors . . . 21
Double-acting click . . . 222
Drag link . . . 204
Drill feed, automatic . . . 267
Driver . . . 7
Driver, least follower for given . . 144

PAGE
Eccentric . . . 199
" rod . . . 193
Eight-day clock . . . 256
Elementary combination . . . 7
Ellipses, compulsory rotation of rolling . . . 60
Ellipses, lobed wheels from . . . 57
" rolling . . . 55
Endless screw . . . 174
Epicyclic trains . . . 268
" " velocity ratio in . . 269
Epicycloid, construction of . . . 76
" and hypocycloid . . . 67
" and pin . . . 69
" and radial line . . . 68
Escapement, anchor . . . 188
" crown-wheel . . . 187
" dead-beat . . . 190
Exact straight-line motion, Peaucellier's . . . 275
Exact straight-line motion, Russell's, 282

Faces and flanks, teeth with both . . 92
Face gearing . . . 49, 159
Faces or flanks only, teeth with . . 87
Face of tooth . . . 85
Feed, automatic drill . . . 267
Ferguson's paradox . . . 271
Flank of tooth . . . 85
Follower . . . 7
" for given driver, least . . . 144
Frequency of contact between teeth . 259
Friction gearing . . . 44
Frictional catch . . . 223

Gearing . . . 45
" bevel . . . 47, 151
" classification of . . . 45
" face . . . 49, 159
" friction . . . 44
" pin . . . 131
" screw . . . 49, 172
" skew . . . 48, 157
" spur . . . 46, 85
" twisted . . . 50, 149
Gears, mitre . . . 48
Gorge, circle of the . . . 36
Grant's odontograph . . . 124
Graphic representation of motion . . 10
Grasshopper motion . . . 283

PAGE
Guide pulleys 233

Hooke's coupling or universal joint . 214
Hour-glass worm 177
Hunting cog 260
Hyperboloid of revolution 34
Hyperboloids, rolling 36
Hyperboloids, velocity ratio of rolling 37
Hyperboloid and cones, analogy between 42
Hypocycloid and epicycloid 67
" construction of 78

Idle wheel 244
Inside screw 173
Instantaneous centre 18
Interchangeable lobed wheels . . . 59
" spur wheels . . . 97
Intermittent motion 2, 62
Involutes 70
Involute, construction of 79
" teeth, interference of . . . 109
" teeth, peculiar properties of 111
" system of teeth 104

Length of belts 236
Line of action 26
" pitch 45
" epicycloid and radial 68
Linear velocity 3
Links 15, 193
Linkwork 193
" aggregate velocities by . . 263
" directional relation in . . 20
" multiplication of oscillation by 211
" quick return motion by . . 213
" rapidly varying velocity by, 212
" circular motion into rectilinear reciprocation and the reverse by 194
" continuous rotation into rotative reciprocation and the reverse by 200
" transmission of continuous rotation by 204
" velocity ratio in 16
Lobed wheels 56

PAGE
Low-numbered pinions 140
Logarithmic spiral 52
" " construction of . . 53
" " lobed wheels from, 56

Machine, definition of 1
Machines, parts of 6
Mangle rack 65
" wheels 63
Mechanism, definition of pure . . . 1
" train of 7, 240
Mitre gears 48
Month clock 257
Motion and rest 2
Motions, composition of 11
Motion, curvilinear 2
" directional relation in contact, 24
" graphic representation of . . 10
" grasshopper 283
" intermittent 2, 62
" kinds of 2
" modes of transmission of, 7, 16
" oscillating 2
" parallel 275
" parallelogram of 11
" Peaucellier's exact straight-line 275
" periodic 5
" polygon of 12
" quick return by linkwork . . 213
" reciprocating 2
" rectilinear 2
" resolution of 13
" resultant 11
" Roberts' approximate straight-line 284
" Russell's exact straight-line . 282
" screw for variable 178
" Tchebicheff's approximate straight-line 285
" velocity ratio in contact . . 21
" vibrating 2
" Watt's approximate straight-line 278
" Watt's sun and planet . . . 272
" Whitworth's quick return . 181

Non-circular wheels 150
Notation 247
Numbers for trains, approximate . . 260

PAGE
Odontograph, Grant's 124
" improved Willis's . . 122
" Robinson's templet . 126
" Willis's 119
Oldham's coupling 185
Oscillating motion 2
Oscillations by linkwork, multiplications of 211
Outside screw 173
Oval chuck 286

Paradox, Ferguson's 271
Parallel motions 275
Parallelogram of motions 11
Parts of machines 6
Path 2
Peaucellier's exact straight-line motion 275
Period 6
Periodic motion 5
Pin and epicycloid 69
Pin and slotted bar 182
Pin and slotted crank 179
Pin gearing 131
Pin gearing, limiting diameter of pins 136
Pin gearing, pins of sensible diameter 133
Pinions 47
" low-numbered 140
" two-leaved 142
Pitch and arcs of action, relation between 95
Pitch circle 45
" circular 80
" diameters, table of 81
" diametral 82
" line 45
" of screw 172
" surface 45
" point 85
Polygon of motions 12
Problems 289
Properties of involute teeth, peculiar 111
Pulley, convexity of 232
" guide 233
" stepped 229
" tightening 231
" Weston's differential . . . 265

PAGE
Quick return motion by linkwork . . 213
" " " Whitworth's . 18

Rack 47
Rack and wheel 98, 110, 137
Ratchet, click and 219
Ratio, velocity 8
Recess, angle and arc of 86
Reciprocating motion 2
Rectification of circular arcs . . . 74
Rectilinear motion 2
Relation, directional 8
Representation of motion, graphic . 10
Resolution of motion 13
Rest and motion 2
Resultant motion 11
Reversible click 220
Revolution 2
" cam for complete 169
" hyperboloid of 34
Roberts' approximate straight-line motion 284
Robinson's templet odontograph . . 126
Rolling cones 29
" contact 14
" " condition of 24
" cylinders 27
" ellipses 55
" ellipses, compulsory rotation of 60
" hyperboloids 36
" hyperboloids, velocity ratio of, 37
Rotation 2
" synchronal 28
Russell's exact straight-line motion . 282

Screw 172
" endless 174
" for variable motion 178
" gearing 49, 172
" inside 173
" outside 173
" pitch of 172
Shifting belts 231
Similarity in all modes of transmission 26
Size of describing circle 94
Skew bevel wheels 157
Skew gearing 48
Sliding contact 14

PAGE
Sliding in contact motions, percentage of 40
Sliding in contact motions, velocity of, 24
Slotted bar and cam 183
" bar and pin 182
" crank and pin 179
Spiral, logarithmic 52
Spiral, logarithmic, lobed wheels from 56
Spur and bevel wheels, relative action of 156
Spur gearing 46, 85
Stepped pulley 229
" wheel 50
Straight-line motion, Peaucellier's exact 275
Straight-line motion, Roberts' approximate 284
Straight-line motion, Russell's exact 282
Straight-line motion, Tchebicheff's approximate 285
Straight-line motion, Watt's approximate 278
Surface, pitch 45
Sun and planet motion, Watt's 272
Synchronal rotations 28

Tappets 165
Tchebicheff's approximate straight-line motion 285
Teeth, approximate forms of 113
" customary dimensions of 102
" definitions of parts of 85
" frequency of contact between, 259
" interference of involute 109
" peculiar properties of involute, 111
" unsymmetrical 148
" with both faces and flanks 92
" with faces or flanks only 87
Templet odontograph, Robinson's 126
Tightening pulleys 231
Tooth outline in pin gearing, derived, 133
" " in bevel gearing 153
Trains, approximate numbers for 260
" directional relation in 243
" epicyclic 268
" examples of clock 251
" of mechanism, definition 7
" for rope-making machinery 273
" method of designing 249

PAGE
Trains, value of 240
" velocity ratio in epicyclic 269
Trammel 285
Transmission of motion, modes of, 7, 16
Transmission, similarity in all modes of 26
Tredgold's method for bevel wheels 154
Twisted belt 232
" gearing 50, 149
Two-leaved pinion 142

Uniform periodic motion 6
" velocity 3
Unsymmetrical teeth 148

Value of a train 240
Variable velocity 4
Velocity, angular 4
" constant or uniform 3
" linear 3
" of sliding in contact motions, 24
" ratio 8
" " condition of constant 24
" " in contact motions 21
" " in epicyclic trains 269
" " in linkwork 16
" " in wrapping connectors 20
" " of rolling hyperboloids 37
" " variable 4
Verge 187
Vibrating motion 2

Watt's approximate straight-line motion 278
Watt's crank substitute, or sun and planet motion 272
Weston's differential pulley 265
Wheel and rack 98, 110, 137
Wheels, annular 47, 101, 138
" bevel 151
" idle 244
" interchangeable 97
" " lobed 59
" lobed 56
" mangle 63
" non-circular 150
" skew bevel 157
" stepped 50

PAGE
Wheel, worm and 174
Whitworth's quick return motion . . 181
Willis' method of approximation by circular arcs 114
Willis' odontograph 119
" " improved . . . 122
Wipers 165

PAGE
Worm and wheel 174
" hour-glass 177
Wrapping-connectors 15, 227
Wrapping connectors, directional relation in 21
Wrapping connectors, velocity ratio in 20

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